Synchronization of coupled augmented Lorenz oscillators with parameter mismatch

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Abstract: We have recently developed a chaotic gas turbine, the rotational motion of which might simulate the random reversals of large-scale circulation in turbulent Rayleigh-Bénard convection. The nondimensionalized equations of motion of the turbine are represented as a star network of many Lorenz subsystems sharing dimensionless angular velocity as the central node, referred to as augmented Lorenz equations. In this study, we investigate the dynamical nature of the augmented Lorenz model, focusing on the chaotic synchronizability of coupled augmented Lorenz oscillators with parameter mismatch.

Key Words: Lorenz equations, augmented Lorenz equations, chaotic gas turbine, synchronization, parameter mismatch

1. Introduction

Motivated by the chaotic waterwheel [1, 2], the rotational motion of which is subject to the Lorenz equations [3], as a historical physical model for the turbulent Rayleigh-Bénard convection of a hydrodynamic flow, we have recently developed a chaotic gas turbine that randomly reverses its direction of rotation [4, 5]. The random reversals of the turbine rotor might simulate those of large-scale circulation, often called mean wind, in actual turbulent Rayleigh-Bénard convection with a high Rayleigh number exceeding $10^6$ and an aspect ratio of order unity [6–8]. Our gas turbine was developed not for industrial use but as a pedagogic machine for learning the Lorenz model. In fact, our machine is driven by the aerodynamic drag generated on the turbine blades, which remains within a limited space around the central axis of the turbine, referred to as the working range, unlike existing gas turbines driven by aerodynamic lift. As a result, the nondimensionalized equations of motion of our turbine are represented as a large-scale system of nonlinear ordinary differential equations, i.e., a star network of infinitely many Lorenz subsystems sharing the variable $X$ as the central node. We refer to the nondimensionalized equations with the number of Lorenz subsystems truncated at a finite number $N$ as augmented Lorenz equations.

The augmented Lorenz equations are specified by two dimensionless parameters that are defined as functions of the mechanical parameters of the turbine. These parameters were shown to be equivalent...
to the reduced Rayleigh number and the Prandtl number [4]. Although the augmented Lorenz model consists of the nondimensionalized equations of motion of the chaotic gas turbine, its applications are no longer limited to the physics of the turbine when the equations are viewed as a general dimensionless dynamical model. Then, the reduced Rayleigh number and the Prandtl number are also no longer dependent on the mechanical structure of the turbine and can instead be set to arbitrary real numbers in accordance with the application at hand.

One such application is expected to be cryptosystems based on the chaotic synchronization of coupled nonlinear oscillators [9, 10]. Chaos-based cryptography is an emerging research field of chaos and nonlinear dynamics. Many methods for chaos-based cryptography have been devised, such as chaotic synchronization [9–11], chaotic shift keying [12], controlling chaos [13–15], distributed dynamics encryption [16, 17], public-key encryption [18] and chaotic block ciphers [19]. For the cryptanalysis of chaos-based cryptosystems, see [20]. Cryptosystems based on chaotic synchronization have actually been implemented using analog electronic circuits subject to the Lorenz model [9, 10], in which the drive and response systems are coupled with each other by sharing the chaotic signal X or Y generated by the Lorenz equations [21, 22]. The Lorenz model is useful for many applications of chaos because its dynamical properties such as the bifurcation structure and basins of attraction have been well investigated [23–26]. In contrast, much is unknown about the dynamical properties of the augmented Lorenz equations. Nevertheless, the augmented Lorenz model may also be applicable to chaotic synchronization, since it is a star network of many Lorenz subsystems and hence is expected to inherit the dynamical nature of the Lorenz model. Exploring the chaotic synchronization of augmented Lorenz oscillators will provide important information for developing real-world applications of the augmented Lorenz model, which is the purpose of this study.

This paper is an extended version of our previous study published in NOLTA2012 [27]. In this paper, we examine the chaotic synchronizability of coupled augmented Lorenz oscillators, focusing on the effect of the parameter mismatch between the augmented Lorenz equations governing the oscillators on the synchronization error. To determine the parameter settings for the equations that give rise to chaos, the bifurcation structure is studied as a function of the reduced Rayleigh number and the number of Lorenz subsystems. The two oscillators are coupled with direct coupling, not with diffusive coupling. It was previously reported that coupled Lorenz oscillators with diffusive coupling are multistable in the sense of the coexistence of perfect synchronization and antisynchronization manifolds, which results in riddled basins of attraction [25, 26]. As will be shown later, by consideration of the symmetry of the augmented Lorenz equations, augmented Lorenz oscillators coupled with direct coupling do not appear to be multistable and may have no riddled basins of attraction. Because of this observation, we may not have to consider the effect of initial conditions on distinct attractors of the augmented Lorenz equations.

This paper is organized as follows. In section 2, we give a brief summary of the mathematics of the augmented Lorenz model and show the direct coupling of augmented Lorenz oscillators. The chaotic synchronizability of coupled oscillators as well as desynchronization due to parameter mismatch are theoretically analyzed in terms of the Lyapunov function, which were not given in our previous paper [27]. The chaotic synchronization between augmented Lorenz oscillators with parameter mismatch should be treated in terms of generalized synchronization. Hence, we refer to the unifying definition of synchronization for coupled dynamical systems that was proposed by Brown and Kocarev [28] and refined by Boccaletti et al. [29]. In section 3, we report numerical experiments that were conducted to estimate the bifurcation structure of the augmented Lorenz equations and assess the synchronizability of the coupled oscillators with parameter mismatch. Our previous numerical results reported in [27] are incorrect in that for the plots in Fig. 2 of [27], the synchronization error is one order smaller than it really is, although the general tendency of the plots is qualitatively correct. As corrections to the previous results, correct estimates are given in this section. In sections 4 and 5, we discuss our results and give conclusions, respectively.
2. Theory

We first give a brief summary of the augmented Lorenz model. For details, see [4]. As shown in [4], the augmented Lorenz model may simulate the dynamical behavior of large-scale circulation in turbulent Rayleigh-Bénard convection with high Rayleigh numbers exceeding $10^6$ and an aspect ratio of unity. In our model, the physical elements that govern the thermal convection, i.e., buoyancy, viscous drag and thermal dissipation, are embedded into the Prandtl number and the reduced Rayleigh number in much the same way as in the Lorenz model. The augmented Lorenz equations as a general dynamical model are given as

\begin{align}
\dot{X} &= \sigma [\text{tr} ((n^{-1})^2 Y) - X], \\
\dot{Y} &= R X - n Z X - Y, \\
\dot{Z} &= n Y X - Z, \\
R &= R_0 n^2 \Phi W, \\
\end{align}

where $X$ is a dimensionless scalar variable (corresponding to the dimensionless angular velocity of a chaotic turbine or the dimensionless velocity field of a convective roll); $Y$ and $Z$ are dimensionless $N \times N$ diagonal matrices whose diagonal components $Y_n$ and $Z_n$ with $n$ running from 1 to $N$ represent dimensionless scalar variables (corresponding to dimensionless spatial Fourier coefficients, truncated at a finite order of $N$; of the air inflow into the turbine), respectively; $\dot{X}$, $\dot{Y}$ and $\dot{Z}$ are the derivatives of $X$, $Y$ and $Z$ with respect to dimensionless time $\tau$; $\text{tr} (\cdot)$ denotes the diagonal sum of a matrix; $\sigma$ and $R_0$ are dimensionless scalar parameters. The dimensionless matrix $R$ is defined using

\begin{align}
R &= R_0 n^2 \Phi W, \\
\Phi &= \text{diag} \left( \phi - \frac{1}{2} \sin 2 \phi, \cdots, \frac{1}{n-1} \sin (n-1) \phi - \frac{1}{n+1} \sin (n+1) \phi, \cdots \right), \\
W &= \text{diag} (\sin \phi, \sin 2 \phi, \cdots, \sin N \phi), \\
\end{align}

where $\phi$ is the working range measured in radians and is an appropriately chosen scalar constant.

When $N = 1$, Eqs. (1)–(3) are exactly equivalent to the Lorenz equations with a geometric parameter of unity. In this sense, $\sigma$ and $R_0$ correspond to the Prandtl number and the reduced Rayleigh number, respectively. Equations (1)–(3) can be viewed as a star network of $N$ Lorenz subsystems sharing $X$ as the central node, as illustrated in Fig. 1(a).

Augmented Lorenz oscillators can be coupled by sharing $X$ or $Y$ between the oscillators, as shown in Figs. 1(b) and (c), respectively. This coupling mode is referred to as direct coupling. Let us consider a drive-response system consisting of directly coupled augmented Lorenz oscillators. When the drive system is subject to Eqs. (1)–(3), the response system is subject to

\begin{align}
X' &= X, \\
Y' &= R' X' - n Z' X' - Y', \\
Z' &= n Y' X' - Z', \\
\end{align}

for direct coupling via $X$ and

\begin{align}
\dot{X}' &= \sigma' [\text{tr} ((n^{-1})^2 Y') - X'], \\
Y' &= Y', \\
\dot{Z}' &= n Y' X' - Z', \\
\end{align}

for direct coupling via $Y$, where $X'$, $Y'$ and $Z'$ denote the variables of the response system, $R'$ includes the reduced Rayleigh number $R'_0$ and $\sigma'$ is the Prandtl number.

Equations (1)–(3) are invariant under the transformation of the variables $(X, Y, Z) \rightarrow (-X, -Z, Z)$, which allows the coexistence of perfect synchronization and antisynchronization. However, neither
direct coupling via $X$ nor direct coupling via $Y$ allows the antisynchronization of the coupled oscillators. Hence, neither the drive-response system with direct coupling via $X$ nor that with direct coupling via $Y$ may be multistable with riddled basins of attraction, unlike coupled Lorenz oscillators with diffusive coupling [26].

We next conduct a dynamical stability analysis of directly coupled augmented Lorenz oscillators with parameter mismatch. The chaotic synchronization between such non-identical nonlinear oscillators is a type of generalized synchronization, for which the unifying definition of synchronization introduced by Brown and Kocarev [28] and refined by Boccaletti et al. [29] is useful. The central idea of their definition is that if neighbors of the trajectories representing the dynamical behavior of the drive oscillator can be mapped into neighbors of those of the response oscillator using an appropriate continuous function, the response oscillator can be said to be synchronized with the drive oscillator.

Let us denote the synchronization errors between the drive and response oscillators as

$$
\begin{align*}
e_1 &= f_r(X') - f_d(X), \\
e_2 &= g_r(Y') - g_d(Y), \\
e_3 &= h_r(Z') - h_d(Z),
\end{align*}
$$

where $e_2$ and $e_3$ are $N$-dimensional diagonal matrices to measure the synchronization errors, with their $n$th diagonal components denoted as $e_{2n}$ and $e_{3n}$, respectively, $f_r$ and $f_d$ are continuous scalar functions, and $g_r$, $g_d$, $h_r$ and $h_d$ are continuous vector functions. In this study, $f_r$, $f_d$, $g_r$, $g_d$, $h_r$ and $h_d$ are set to the identity functions as

$$
\begin{align*}
e_1 &= X' - X, \\
e_2 &= Y' - Y, \\
e_3 &= Z' - Z,
\end{align*}
$$

These settings are useful for assessing the synchronizability of coupled augmented Lorenz oscillators with parameter mismatch in comparison with that of coupled identical oscillators, as will be shown in the following.

In the case of direct coupling via $X$, from Eqs. (1)–(3) and Eqs. (7)–(9), we obtain

$$
\begin{align*}
e_1 &= 0, \\
\dot{e}_2 &= (R' - R)X - ne_3X - e_2, \\
\dot{e}_3 &= ne_2X - e_3,
\end{align*}
$$

that is, for $n = 1, \ldots, N$,

$$
\begin{align*}
e_1 &= 0, \\
\dot{e}_{2n} &= (R_0' - R_0)n^2\phi_nW_nX - nXe_{3n} - e_{2n}, \\
\dot{e}_{3n} &= nXe_{2n} - e_{3n},
\end{align*}
$$

Fig. 1. Augmented Lorenz oscillators. (a) Decoupled oscillators, (b) coupled oscillators sharing $X$ (direct coupling via $X$) and (c) coupled oscillators sharing $Y$ (direct coupling via $Y$).
where $\phi_n$ and $W_n$ are the $n$th diagonal components of $\Phi$ and $W$, respectively. Let us introduce the Lyapunov function $E$ defined as

$$E = \frac{1}{2} \left[ e_1^2 + \sum_{n=1}^{N} (e_{2n}^2 + e_{3n}^2) \right].$$

From Eqs. (22)–(24), the derivative of $E$ with respect to dimensionless time is obtained as

$$\dot{E} = e_1 \dot{e}_1 + \sum_{n=1}^{N} (e_{2n} \dot{e}_{2n} + e_{3n} \dot{e}_{3n}),$$

$$= \sum_{n=1}^{N} \left[ (R_0 - R_0) n^2 \phi_n W_n e_{2n} - e_{3n}^2 - e_{2n}^2 \right].$$

When there is no parameter mismatch, i.e., $\Delta R = R_0' - R_0 = 0$, then $\dot{E} \leq 0$ and $\dot{E} = 0$ if and only if $e_2 = e_3 = 0$. Accordingly, the fixed point at $e_1 = 0$, $e_2 = 0$ and $e_3 = 0$ is asymptotically stable. Hence, the chaotic synchronization is asymptotically achieved. In contrast, when $\Delta R \neq 0$, $e_{2n}$ includes the term with $O(\Delta R)$, which is apparent from the right-hand side of Eq. (23). This implies that $e_2$ has a considerable contribution proportional to the parameter mismatch $\Delta R$. Since the right-hand side of Eq. (24) includes the term proportional to $e_{2n}$, the synchronization error $e_3$ is also predicted to have a considerable contribution proportional to $\Delta R$. Note that the parameter mismatch $\Delta \sigma = \sigma' - \sigma$ does not affect the synchronizability of the oscillators directly coupled via $X$.

Likewise, in the case of direct coupling via $Y$, from Eqs. (1)–(3) and Eqs. (10)–(12), we obtain

$$\dot{e}_1 = (\sigma' - \sigma) \{ \text{tr}[(n^{-1})^2 Y] \} - \sigma' X' + \sigma X,$$

$$e_2 = 0,$$

$$\dot{e}_3 = nY e_1 - e_3.$$

When there is no parameter mismatch in $\sigma$, i.e., $\Delta \sigma = 0$, Eq. (27) is rewritten as $\dot{e}_1 = -\sigma e_1$. Thus, $e_1 \rightarrow 0$ and $e_3 \rightarrow 0$ as $\tau \rightarrow \infty$. This means that the two identical oscillators directly coupled via $Y$ can asymptotically synchronize with each other.

When $\Delta \sigma \neq 0$, the right-hand side of Eq. (27) consists of three terms, i.e., the first term proportional to $\Delta \sigma$ and the second and the third terms $-\sigma' X' + \sigma X$. The first term generates a synchronization error proportional to $\Delta \sigma$ in $e_1$, whereas the second and the third terms, which are approximately equal to $-X \Delta \sigma$ when $X' \approx X$, give rise to a synchronization error asymmetrical around $\Delta \sigma = 0$ in $e_1$. Since the right-hand side of Eq. (29) includes the term proportional to $e_1$, the synchronization error $e_3$ is predicted to have a considerable amount proportional to $\Delta \sigma$ in $e_1$. Nevertheless, unlike the $X$-coupling case, $e_3$ is also predicted to be asymmetrical around $\Delta \sigma = 0$ because of the terms $-\sigma' X' + \sigma X$ included in the right-hand side of Eq. (27). The parameter mismatch $\Delta R$ does not affect the synchronizability of the oscillators directly coupled via $Y$.

3. Numerical analysis

In the numerical analysis of the bifurcation structure of the augmented Lorenz equations, the working range $\phi$ and Prandtl number $\sigma$ were fixed at $\phi = 0.36$ [rad] and $\sigma = 28.3$, respectively. These parameter settings are known to generate the chaotic motion of the chaotic gas turbine [4, 5]. Bifurcation diagrams of the augmented Lorenz model were estimated as functions of $R_0$ and $N$ by numerically integrating Eqs. (1)–(3) using the fourth-order Runge-Kutta method with a dimensionless time width of $4 \times 10^{-5}$ for $0 \leq R_0 \leq 3500$ and $1 < N \leq 1000$. The minimum difference between adjacent values of $R_0$ was set to 1. The initial value of $X$ was given as $X(0) = 0$ and those of $Y_n$ and $Z_n$ were given as Gaussian random numbers with mean 0 and variance 1. The initial 250 000 numerical solutions were discarded to eliminate the initial transient solutions, which are strongly dependent on the initial conditions.
As an example, Fig. 2 shows the estimated bifurcation diagram as a function of $R_0$ at $N = 100$, which represents the cross-sectional plots satisfying $-0.01 \leq Y_{10} \leq 0.01$. The general features of the bifurcation diagram are qualitatively similar to those of the Lorenz model, despite some minor differences. The threshold of $R_0$ for the onset of chaos was found to shift from $\sim 500$ to $\sim 1200$ as $N$ increases from 5 to 6 and reaches $\sim 1500$ at $N = 100$. A window, wherein chaotic oscillations are superseeded by nonchaotic oscillations, appears between $\sim 1600$ and $\sim 1800$ when $N$ exceeds 350. Chaos appears to be well developed at $R_0 \geq 2000$. The parameter settings of $\phi = 0.36$ [rad], $\sigma = 28.3$ and $N = 100$ can generate chaotic oscillations useful for applications of the augmented Lorenz model. Under such parameter settings, the chaotic oscillation of the augmented Lorenz oscillator is well developed around $R_0 = 3000$, as shown in Fig. 2.

We next observed the synchronization errors between two augmented Lorenz oscillators coupled with direct coupling via $X$ as functions of the parameter mismatch in $R_0$. In this coupling mode, the synchronization errors are induced only by the parameter mismatch $\Delta R$, not by $\Delta \sigma$. The relative parameter mismatch in $R_0$ is defined as $r = \Delta R / R_0$. At each $r$ increasing from $-0.1$ to $0.1$ around $R_0 = 3000$ with an increment of 0.01, Eqs. (1)–(3) and Eqs. (7)–(9) with $\phi = 0.36$ [rad], $\sigma = 28.3$ and $N = 100$ were numerically integrated using the fourth-order Runge-Kutta method with a dimensionless time width of $4 \times 10^{-5}$. The initial 450 000 solutions were discarded to eliminate the initial transient part of the synchronization process. Then, the synchronization errors were estimated for $T = 50$ 000 solutions of $Z_n$ and $Z'_n$ at each $r$. It is convenient to use the mean synchronization error $E_X(r)$ as a function of $r$, defined as

$$E_X(r) = \frac{1}{T} \sum_{t=1}^{T} \left( \frac{1}{N} \sum_{n=1}^{N} e^2_{3n}(t, r) \right) = \frac{1}{T} \sum_{t=1}^{T} \left( \frac{1}{N} \sum_{n=1}^{N} (Z'_n(t, r) - Z_n(t))^2 \right). \quad (30)$$

Equation (30) measures the root-mean-squared synchronization error between $Z_n$ and $Z'_n$ that are the subsets of the variables representing the coupled augmented Lorenz oscillators. It allows us to assess the synchronizability in the sense of almost synchronization, as reported in [29]. Our method for estimating the synchronization error suffices for the purpose of this study, for instance, for comparing the synchronizability between direct coupling via $X$ and that via $Y$. Equation (30) also estimates the synchronization error over long-time segments of the dynamical behaviors in accordance with our theoretical analysis indicating that the chaotic synchronization between two identical augmented Lorenz oscillators is asymptotically achieved.

Estimates of $E_X(r)$ are shown in Fig. 3. The mean synchronization error is estimated to be $E_X(0) \approx 0.16$ at $r = 0$ and $E_X(-0.1) = E_X(0.1) \approx 77.98$ at $r = -0.1$ and 0.1. The increment in $E_X(r)$ is linear with respect to $|r|$ and symmetrical around $r = 0$.

We performed a similar observation of the synchronization errors for direct coupling via $Y$ as functions of the parameter mismatch in $\sigma$. In this case, the synchronization errors are induced only by $\Delta \sigma$, not by $\Delta R$. The relative parameter mismatch in $\sigma$ is defined as $s = \Delta \sigma / \sigma$. The remaining parameters were fixed at $\phi = 0.36$ [rad], $R_0 = 3000$ and $N = 100$. At each $s$ increasing from $-0.1$ to 0.1 around $\sigma = 28.3$ with an increment of 0.01, Eqs. (1)–(3) and Eqs. (10)–(12) were numerically integrated using the fourth-order Runge-Kutta method with a dimensionless time width of $4 \times 10^{-5}$. The initial 450 000 solutions were discarded to eliminate the initial transient part. The synchronization errors were estimated for $T = 50$ 000 solutions of $Z_n$ and $Z'_n$ at each $s$. In much the same way as Eq. (30), the mean synchronization error $E_Y(s)$ is defined as

$$E_Y(s) = \frac{1}{T} \sum_{t=1}^{T} \left( \frac{1}{N} \sum_{n=1}^{N} e^2_{3n}(t, s) \right) = \frac{1}{T} \sum_{t=1}^{T} \left( \frac{1}{N} \sum_{n=1}^{N} (Z'_n(t, s) - Z_n(t))^2 \right). \quad (31)$$

Estimates of $E_Y(s)$ are shown in Fig. 4. The mean synchronization error is estimated to be $E_Y(0) \approx 0.46$ at $s = 0$, $E_Y(-0.1) \approx 18.06$ at $s = -0.1$ and $E_Y(0.1) \approx 16.24$ at $s = 0.1$. The increment in $E_Y(s)$ is linear with increasing $|s|$, although it is asymmetrical around $s = 0$ in contrast to $E_X(r)$ for direct coupling via $X$. Such asymmetrical estimates agree with our theoretical prediction.
Fig. 2. Bifurcation diagram as a function of $R_0$ at $N = 100$ ($\phi = 0.36$ [rad] and $\sigma = 28.3$).

Fig. 3. Mean synchronization error $E_X(r)$ as a function of rate of parameter mismatch $r$ for direct coupling via $X$. $R_0 = 3000$ ($\phi = 0.36$ [rad], $N = 100$ and $\sigma = 28.3$).

Fig. 4. Mean synchronization error $E_Y(s)$ as a function of rate of parameter mismatch $s$ for direct coupling via $Y$. $\sigma = 28.3$ ($\phi = 0.36$ [rad], $N = 100$ and $R_0 = 3000$).
4. Discussion

The general features of the bifurcation structure of the augmented Lorenz model are qualitatively similar to those of the Lorenz model. The augmented Lorenz model, as a star network of many Lorenz subsystems, basically inherits the dynamical nature of the Lorenz model. In fact, as has been shown in the preceding sections, chaotic synchronization is asymptotically achieved by the coupled augmented Lorenz oscillators sharing \( X \) or \( Y \) in much the same way as by directly coupled Lorenz oscillators. This has been verified both theoretically and numerically. Our theoretical analysis based on the Lyapunov function for the dynamical stability of the synchronization errors indicates that the fixed point at \( e_1 = 0 \) and \( e_2 = e_3 = 0 \) is asymptotically stable when there is no parameter mismatch, while the synchronous state loses its stability and the synchronization errors linearly increase with increasing parameter mismatch when there is parameter mismatch. In section 3, we have shown estimates of \( E_X(0) \approx 0.16 \) and \( E_Y(0) \approx 0.46 \), despite no parameter mismatch. These estimates are considered to reflect the asymptotical stability of the synchronization manifold and will decrease as \( T \) increases. Furthermore, in section 2, we have shown that the synchronization error is symmetrical around \( \Delta R = 0 \) for direct coupling via \( X \) and asymmetrical around \( \Delta \sigma = 0 \) for direct coupling via \( Y \). These predictions have also been numerically verified by estimating the root-mean-squared synchronization error as a function of the relative parameter mismatch in the numerical simulations.

In applications of the augmented Lorenz model such as cryptosystems based on chaotic masking and chaotic synchronization, augmented Lorenz oscillators would be required to generate chaos with a sufficient degree of complexity. To meet this requirement, augmented Lorenz oscillators with \( N = 100 \) would suffice. In real-world applications, the parameter mismatch between the drive and response oscillators is inevitable when the oscillators are implemented as hardware systems. Our results indicate that for direct coupling via \( X \), the rate of increase in the synchronization error is symmetrical around \( \Delta R = 0 \), while for direct coupling via \( Y \), the rate is asymmetrical \( \Delta \sigma = 0 \). Hence, direct coupling via \( Y \) is less tractable than that via \( X \) in applying coupled augmented Lorenz oscillators to cryptosystems based on chaotic synchronization. Instead, cryptosystems using direct coupling via \( Y \) have the remarkable virtue that the multiplex masking of messages is feasible by using each \( Y_n \) as the carrying signal of the message \( m_n \), i.e., \( Y_n + m_n \) where \( 1 \leq n \leq N \). These results will be useful for designing the tolerance of a cryptosystem to parameter mismatch as well as for determining the minimal difference between adjacent encryption-decryption keys in the \((R_0, \sigma)\)-key space.

5. Conclusions

We have shown the augmented Lorenz equations to be a general dimensionless dynamical model and theoretically analyzed the chaotic synchronizability of coupled nonlinear oscillators subject to the augmented Lorenz model. The direct coupling of the oscillators sharing \( X \) or \( Y \) is suggested to induce no synchronization manifolds representing multistability, i.e., no riddled basins of attraction, by consideration of the symmetry of the augmented Lorenz equations. Theoretical analysis based on the Lyapunov function indicates that the chaotic synchronization of directly coupled augmented Lorenz oscillators can be asymptotically achieved and that the parameter mismatch between the oscillators induces a synchronization error that grows linearly with the magnitude of the mismatch.

In numerical experiments, we have estimated the bifurcation structure of the augmented Lorenz model as a function of the reduced Rayleigh number \( R_0 \) and the number of Lorenz subsystems \( N \). The bifurcation structure is found to be qualitatively similar to that of the Lorenz model. The mean-squared synchronization error between directly coupled augmented Lorenz oscillators has been estimated as a function of the relative parameter mismatch in \( R_0 \) around \( R_0 = 3000 \) or the relative parameter mismatch in \( \sigma \) around \( \sigma = 28.3 \). The estimates of the mean synchronization error agree with our theoretical predictions.

The application of the augmented Lorenz model to real-world problems is an open question to be investigated in future studies. One such application is cryptosystems based on chaotic synchronization. However, the present study has shown that a small parameter mismatch between coupled augmented Lorenz oscillators would cause a large synchronization error that would prevent the hardware implementation of the cryptosystems. Recently, we have found a new method for applying the augmented...
Lorenz model to cryptosystems without resorting to chaotic synchronization, which can be realized by generalizing the augmented Lorenz model. Details of this method will be reported in a future paper.

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