Invited Paper

Emergence of synchronization in bio-inspired memristor-coupled oscillatory cells

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Abstract: The nanoscale memristor is a serious candidate to become the core element of novel ultra-high density low-power non-volatile memories and innovative pattern recognition systems based upon oscillatory associative and dynamic memories. Furthermore, this peculiar device also has the potential to capture the behavior of a biological synapse more efficiently and accurately than any conventional electronic emulator since it exhibits the unique capability of performing computation and storing data at the same physical location and at same time. In addition, it has a flux-controlled conductance which is analogous to the ionic flow-controlled synaptic weight. This chapter gives some insight into the mechanisms underlying the emergence of synchronization between two oscillatory cells coupled through an ideal memristor. The investigations show that in some cases the nonlinear dynamics of the memristor play a key role in the development of synchronous oscillations in the two oscillators. This work sheds light on some aspects of the nonlinear behavior of the still largely unexplored memristor, which is doomed to make an impact in integrated circuit design in the years to come.

Key Words: memristor, oscillatory networks, Hindmarsh-Rose neuron, nonlinear dynamics, synchronization

1. Introduction

Networks of nonlinear and dynamical cells [1] constitute a suitable framework for the modeling of various phenomena arising in interdisciplinary fields ranging from Biology to Chemistry, from Engineering to Physics. Interestingly, the architecture of neuromorphic circuits [2], bio-inspired electronic systems which mimic spatio-temporal pattern recognition processes and neural signal processing mechanisms...
occurring in the brain, is based upon networks of nonlinear and dynamical cells.

Mathematical models describing neuromorphic networks are large sets of nonlinear functional differential equations [3] (or of nonlinear differential inclusions [4], which may be introduced in case the equations are discontinuous) including inputs and random parameters if necessary.

To draw a complete picture of the global dynamical behavior of a system of this kind, it would be mandatory to determine number, local stability and domains of attraction of all attractors, both the stationary (finite equilibria, equilibria at infinity) and non-stationary (limit cycles, quasi-periodic and chaotic steady-state trajectories) ones. Due to the complexity of the mathematical models, fulfilling this task is computationally extensive. This explains why in the literature one may find a limited number of works studying the nonlinear dynamics of neuromorphic networks. Further, most investigations are based on numerical simulations, since carrying out analytical work on these types of models is complicated.

A large deal of efforts has been traditionally devoted to the analysis of non-stationary steady-state behaviors in neuromorphic networks, since these dynamics may reproduce spatio-temporal nonlinear phenomena in Neuroscience. One of the most important aspects of these research endeavors concerns synchronization [5], i.e. a scenario where all the cells of a network act in unison. Some insight into the mechanisms underlying the process of synchronization, especially regarding the influence of the coupling strength on the emergence of this phenomenon, have been gained for networks of distinct topology under various local or global connection layouts. Conventional electronic implementations of the couplings among neuron cells in neuromorphic networks lack the plasticity characterizing biological synapses.

The memristor, a two-terminal circuit element characterized by a nonlinear relation between the time integrals of current and voltage, was theoretically envisioned by Prof. L. O. Chua back in 1971 [6]. Features of the memristor proposed by Prof. Chua were found in a nano-scale film based on Titanium Dioxide in 2008 by a team of Hewlett Packard researchers led by Senior Fellow S. Williams [7]. This discovery has been recently followed by the experimental observation of some aspects of memristor behavior in other nano-structures [8–16]. Some of these physical devices [17–19] are capable to reproduce the nonlinear dynamics of neural synapses with high level of accuracy: they may process and store information at the same time, they may occupy nano-scale volumes, they may be arranged on multi-layer cross-bar array configurations ideally suited for parallel processing, they may consume very little power, and, most importantly, they may exhibit flux-controllable conductances reminiscent of the ion flow-tunable weights of neural synapses. The modulation of the synaptic weights in neural networks form the basis of various biological phenomena such as the learning process. As a result, the availability of an efficient emulator of synaptic behavior is fundamental to mimic the functionalities of neural networks. This is the reason why so much research is currently under way to unfold the full potential of memristors as synaptic couplings among neuron oscillators. Particularly, it is of great interest to explore the nonlinear behaviors including the synchronization phenomena developing in bio-inspired networks of memristor-coupled oscillatory cells, and to develop analytical and numerical methodologies to gain a better understanding of the mechanisms at the origin of these dynamics [20–24].

The aim of this work is to study some aspects of the synchronization properties of two Hindmarsh-Rose (HR) neuron oscillators [25] coupled through an ideal memristor [26]. The results of the investigation highlight the crucial role the memory capability inherent in the memristor synapse plays in the development of synchronization between the two neuron cells. The work, based on numerical simulations, further elucidates the different dynamics of a memristor compared to an ordinary linear resistor by drawing a comparison between the synchronization properties of the two memristor-coupled oscillators and of an equivalent circuit where the memristor is replaced by a suitable resistor. For simplicity throughout this chapter we shall call the two memristor-coupled HR neurons as Memristor Network (MN) and the two resistor-coupled HR neurons as Diffusive Network (DN).

The manuscript is organized as follows: Section 2 presents the MN under study. Section 3 introduces the measure for synchronization adopted in this work, namely the so-called Event Synchronization (ES) [27]. Section 4 sheds light on the mechanisms lying behind the emergence of synchronization
2. Model for two memristor- and resistor-coupled HR neuron cells

According to the model originally proposed by Hindmarsh-Rose (HR) \cite{25}, the dimensionless normalized nonlinear ordinary differential equation (ode) system governing the dynamical behavior of the third-order state vector of a single neuron oscillator are given by

\[
\frac{dx}{d\tau} = y - ax^3 + bx^2 - z + I
\]  
(1)

\[
\frac{dy}{d\tau} = c - dx^2 - y
\]  
(2)

\[
\frac{dz}{d\tau} = r[s(x - x_0) - z],
\]  
(3)

where \( \tau = \frac{t}{t_0} \) denotes the dimensionless normalized time variable, while \( a, b, c, d, r, s \in \mathbb{R}_+ \) are dimensionless constants. The three dimensionless normalized state variables are \( x = \frac{x}{V_0} \), where \( v \) indicates the membrane potential, \( y = \frac{i}{i_0} \), with \( i \) signifying the current flow of Sodium \( Na \) and Potassium \( K \) ions, and \( z = \frac{z}{V_0} \), in which \( i' \) represents the much slower current flow of other types of ions. Using the data reported in one of the pioneering works from Hindmarsh and Rose \cite{28}, the voltage, current and time normalization factors are set to \( V_0 = 1 mV \), \( i_0 = 1 nA \), and \( t_0 = 1 s \) respectively. In Eq. (3) \( x_0 \) expresses the value of state variable \( x \) for the leftmost equilibrium of second-order system of Eqs. (1)–(2) with \( z = 0 \) and \( I = 0 \) (see page 96 in \cite{25}), while \( I \) stands for the dimensionless normalized external current \( i_{ext} \), i.e. \( I = \frac{i_{ext}}{i_0} \).

Let us choose values for \( a, b, c, d, r, s \) and \( x_0 \) as reported in \cite{29}, where some aspect of the dynamics of isolated and diffusively coupled HR neurons was thoroughly investigated. With \( a = 1, b = 3, c = 1, d = 5, r = 0.006, s = 4, \) and \( x_0 = -1.6 \), the isolated HR neuron oscillator may experience a large variety of nonlinear dynamical behaviors subject to the value of \( I \), including stationary state \( (I = 1) \), various types of bursting behavior e.g. one-spike-per-burst \( (I = 1.4) \) and two-spikes-per-burst \( (I = 1.8) \), chaos \( (I = 3) \), and distinct kinds of spiking behavior e.g. period-one \( (I = 3.45) \) and period-two \( (I = 4) \).

Coupling two HR neuron oscillators of the kind described in Eqs. (1)–(3) on the first state variable through a memristor characterized by state \( x \), the modeling equations of the resulting MN have the following form

\[
\frac{dx_i}{d\tau} = y_i - ax_i^3 + bx_i^2 - z_i + I_i - C(x)(x_i - x_j)
\]  
(4)

\[
\frac{dy_i}{d\tau} = c - dx_i^2 - y_i
\]  
(5)

\[
\frac{dz_i}{d\tau} = r[s(x_i - x_0) - z_i], \quad (i, j) \in \{(1, 2), (2, 1)\}
\]  
(6)

where \( C(x) \) represents the effective coupling strength, which depends on the time history of the memristor state. In this work we assume that the synapse connecting the two neurons is an ideal memristor \cite{26}, characterized by a Piece-Wise Linear (PWL) charge \( q \)-flux \( \varphi \) nonlinearity \cite{30} expressed by

\[
q(\varphi) = \beta \varphi + \frac{\alpha - \beta}{2}(|\varphi + \varphi_c| - |\varphi - \varphi_c|),
\]  
(7)

where \( \alpha \) and \( \beta \) (in this work \( \beta > \alpha \in \mathbb{R}_+ \) by hypothesis) are the slopes of the \( q - \varphi \) characteristic and have the measurement units of \( S \), while \( \varphi_c \) denotes the magnitude of the flux at which the nonlinearity presents a point of non-differentiability (usually named as critical flux) and is given in \( V s \). The linear ordinary differential equation regulating the rate of change of the memristor state is

\[
\frac{dx}{d\tau} = x_1 - x_2,
\]  
(8)

where the dimensionless normalized memristor state is defined as \( x = \frac{x}{V_0} \), in which \( \varphi_0 = v_0 t_0 = 1 mV s \) indicates the flux normalization factor, and the dimensionless normalized voltage across the synapse

\[
\frac{dV}{d\tau} = \frac{i}{i_0}.
\]  
(9)

between the two memristor-coupled HR oscillators. Conclusions are outlined in section 5.
is simply the difference between \( x_1 \) and \( x_2 \). The memductance of an ideal flux-controlled memristor is expressed by \( W(\varphi) = \frac{dq}{dx} \). Let us define the dimensionless normalized memductance as

\[
\hat{W}(x) = \hat{\beta} + \frac{\hat{\alpha} - \hat{\beta}}{2} (\text{sgn}(x + \hat{\varphi}_c) - \text{sgn}(x - \hat{\varphi}_c)),
\]

where \( \hat{\alpha} = \alpha R_0 \) and \( \hat{\beta} = \beta R_0 \) are the dimensionless normalized slopes of the PWL charge-flux nonlinearity (7) (the resistance normalization factor is numerically equal to \( R_0 = \frac{\mu_0}{2} = 1 \text{ M}\Omega \)), while \( \hat{\varphi}_c = \frac{\varphi}{\varphi_0} \) represents the dimensionless normalized magnitude of the critical flux \( \varphi_c \). It follows that the effective coupling strength in (4) may be written as

\[
C(x) = \varepsilon \hat{W}(x) = \varepsilon \left( \hat{\beta} + \frac{\hat{\alpha} - \hat{\beta}}{2} (\text{sgn}(x + \hat{\varphi}_c) - \text{sgn}(x - \hat{\varphi}_c)) \right),
\]

where \( \varepsilon \) denotes the coupling strength coefficient. From Eq. (8) it follows that, provided the two oscillators act in synchrony i.e. \( x_1(t) = x_2(t) \) for all times following some transitory behavior, the memristor state approaches an equilibrium point, which may lie anywhere in \( \mathbb{R} \), and, correspondingly, the memductance \( W(\varphi) \) may assume either the off-state value of \( \alpha \) or the on-state value of \( \beta \), attaining the low or high conductance level while \( x \) is still converging to the equilibrium.

Section 3 introduces the measure for synchronization adopted in this work. Then section 4 investigates the synchronization properties of the MN modeled by Eqs. (4)–(6), where the synaptic coupling between the two Hindmarsh-Rose neurons is implemented through the ideal flux-controlled memristor characterized by the PWL \( q-\varphi \) characteristic defined in Eq. (7).

In order to give further insight into the peculiar behavior of memristors in comparison to linear resistors, it is instructive to analyze also the synchronization mechanisms of an equivalent DN where the two neuron oscillators are connected by means of a classical resistor. The equations modeling the DN are still expressed by (4)–(6), in which, however, the effective coupling strength is a constant \( C \), named simply as coupling strength.

### 3. Event synchronization as measure for synchronization

In order to study the synchronization properties of the two coupled neurons, we use the concept of Event Synchronization (ES) [27]. Let us discuss this measure for synchronization in some detail.

In our case the events will be the occurrences of spikes in the steady-state time series of states \( x_1 \) and \( x_2 \), the variables directly involved in the coupling between the two neurons.

Let us briefly review the notion of ES [27]. With reference to two time series \( x \) and \( y \), where \( n_x \) and \( n_y \) events respectively occur at times \( t^x_i \) (\( i = 1, \ldots, n_x \)) and \( t^y_j \) (\( j = 1, \ldots, n_y \)), the quantity

\[
c(x|y) = \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} J^x_{ij},
\]

where

\[
J^x_{ij} = \begin{cases} 1 & \text{if } 0 < t^x_i - t^y_j \leq \tau_{ij}, \\ \frac{1}{2} & \text{if } t^x_i = t^y_j, \\ 0 & \text{otherwise}, \end{cases}
\]

represents the number of times an event appears in \( x \) shortly after it appears in \( y \). “Shortly” here means that the time delay between the occurrences of an event in the two time series may not exceed a specific time delay \( \tau_{ij} \). The choice of this time delay may depend on the problem under study, but nevertheless it must be chosen smaller than half the minimum between the inter-event time intervals in \( x \) and the inter-event time intervals in \( y \) so as to avoid double counting in situations where, for instance, two events in \( x \) are close to an event in \( y \). In mathematical form \( \tau_{ij} \) must satisfy the following condition:

\[
\tau_{ij} \leq \frac{\min\{t^x_{i+1} - t^x_i, t^x_{i} - t^y_{i-1}, t^y_{i+1} - t^y_{i}, t^y_{i} - t^y_{i-1}\}}{2},
\]

295
Moreover, a global time scale \( \tau \), in place of the local ones \( \tau_{ij} \), can be considered, as well. In this case, condition (13) becomes
\[
\tau \leq \min_{i=1,\ldots,n_x, j=1,\ldots,n_y} \left\{ \frac{t^x_{i+1} - t^x_i, t^x_{i} - t^x_{i-1}, t^y_{i+1} - t^y_i, t^y_{i} - t^y_{i-1}}{2} \right\}.
\]

Roughly speaking, \( \tau \) is the maximum time-lag we admit to consider the two signals as synchronized.

In the following, we suppose, without lack of generality, to consider a global time scale \( \tau \), in place of the local ones \( \tau_{ij} \). Thus, in this case we get
\[
J_{ij}^x = \begin{cases} 1 & \text{if } 0 < t^x_i - t^x_j \leq \tau, \\ \frac{1}{2} & \text{if } t^x_i = t^x_j, \\ 0 & \text{otherwise}. \end{cases}
\]

Analogously, it is possible to define the quantity
\[
c(y|x) = \sum_{j=1}^{n_y} \sum_{i=1}^{n_x} J_{ji}^y,
\]
where
\[
J_{ji}^y = \begin{cases} 1 & \text{if } 0 < t^y_j - t^x_i \leq \tau, \\ \frac{1}{2} & \text{if } t^y_j = t^x_i, \\ 0 & \text{otherwise}, \end{cases}
\]
as the number of times an event appears in \( y \) shortly after it appears in \( x \).

Finally, the quantity
\[
Q = \frac{c(x|y) + c(y|x)}{\sqrt{n_x n_y}},
\]
assuming values within \([0,1]\), measures the synchronization of the events, and it is equal to 1 if and only if the events of the two signals are fully synchronized [27].

In fact, in order to have full synchronization, an equal number of events to compare is needed (so \( n_x = n_y = n \)) and only these configurations are admitted:

1. all the events are coincident. Then \( t^x_i = t^y_i \) \( \forall \ i = 1, \ldots, n \). Thus, in this case \( c(x|y) = n/2 \), \( c(y|x) = n/2 \) and \( Q = \frac{n/2 + n/2}{\sqrt{n (n-1)}} = 1 \);

2. the events of a signal always precede those in the second one, but always with a time-lag smaller than \( \tau \). Clearly, in this case we can have two possible configurations:
   a. \( y \) precedes always \( x \). Then, \( 0 < t^x_i - t^y_j \leq \tau \) \( \forall \ i = 1, \ldots, n \), so \( c(x|y) = n, c(y|x) = 0 \) and \( Q = \frac{n+0}{n} = 1 \);
   b. \( x \) precedes always \( y \). Then, \( 0 < t^y_j - t^x_i \leq \tau \) \( \forall \ i = 1, \ldots, n \), so \( c(x|y) = 0, c(y|x) = n \) and \( Q = \frac{0+n}{n} = 1 \);

3. \( k \) events (with \( 0 < k < n \)) of \( y \) precede \( k \) events of \( x \) with a time-lag smaller than \( \tau \), while \( n-k \) events of \( x \) precede \( n-k \) events of \( y \) with a time-lag smaller than \( \tau \), respectively. Thus, \( c(x|y) = k, c(y|x) = n-k \), and \( Q = \frac{k+(n-k)}{n} = 1 \). Let us remark that the choice of \( \tau \) according to condition (14) permits to avoid double counting.

Therefore, it follows that the condition \( Q = 1 \) is equivalent to ask that
\[
|t^x_i - t^y_j| \leq \tau \quad \forall \ i = 1, \ldots, n,
\]
holds, that is
\[
\max_{i=1,\ldots,n} |t^x_i - t^y_i| \leq \tau.
\]

In what follows, we have thus considered condition (19) as indicator of full synchronization.
In particular, we have considered $\tau = 2 \min \{ \Delta t_k \}$ ($k = 1, 2, \ldots, K$), where $\Delta t_k$ is the $k$th time step of the adaptive time step-based ordinary differential equation (ode) solver employed for the numerical integration of (4)–(6) and (8), and $K$ is the total number of time steps in the simulation. We have made this choice, since we are interested in the full synchronization and we admit only a small error in the numerical integration of our equations.

4. Synchronization properties of the MN

Before applying the concept of ES to the bio-inspired circuit described in section 2, let us point out that in this work we do not aim at modeling neural learning rules, such as Spike-Timing-Dependent-Plasticity (STDP) or Spike-Rate-Dependent-Plasticity (SRDP), but rather at grasping a better understanding of the influence of memristor dynamics in the process of synchronization between two HR oscillatory cells. In future works we shall extend our investigations to the mechanisms underlying the process of learning in neuromorphic networks.

Let us consider two identical HR neurons with parameters $a, b, c, d, r, s, x_0$ set to the values reported in section 2 and used in [29]. The initial conditions on the states of the two dynamical systems are taken as $(x_1(0), y_1(0), z_1(0)) = (x_0 - \kappa, c - dx_0, 0)$ and $(x_2(0), y_2(0), z_2(0)) = (x_0 + \kappa, c - dx_0, 0)$ respectively, where $\kappa$, set to 1, makes sure the two neurons are out of phase at the start of the simulations (with $\kappa = 0$ the common initial condition corresponds to the real equilibrium point of Eqs. (4)–(6) under $J = 0$). Using the measure for synchronization reviewed in section 3, the analysis to follow shall study the synchronization properties of the MN described by Eqs. (4)–(6) on the plane spanned by the coupling strength coefficient $\varepsilon$ and by the initial condition on the memristor state $x(0)$. Unless stated otherwise, the initial state of the memristor shall be varied in $[-1.5, 1.5]$ with step $\Delta x(0) = 0.25$. This range of $x(0)$ includes $\pm \dot{\phi}_c$, where, as explained in section 2, $\dot{\phi}_c$ is the dimensionless normalized magnitude of the critical flux.

Further, in order to highlight the crucial influence of the time history of the voltage across the memristor synapse on the emergence of synchrony between the HR neurons, for each synchronization phenomenon in the MN, the behavior of an equivalent DN where the coupling resistor assumes the steady state value of the resistance of the memristor in the MN is also investigated and compared with the dynamics observed in the MN.

In general in this work we let $\phi_c = 1 \text{mV s}$, $\alpha = 10^{-5} \text{S}$ and $\beta = 10^{-3} \text{S}$. Since $\phi_0 = 1 \text{mV s}$ and $R_0 = 1 \text{M}\Omega$ we have $\hat{\phi}_c = 1$, $\hat{\alpha} = 10$ and $\hat{\beta} = 10^3$. Except for special cases, in the MN $\varepsilon$ is varied from $\varepsilon_1 = 10^{-5}$ to $\varepsilon_f = 10^{-3}$ with step $\Delta \varepsilon = 5 \cdot 10^{-5}$. Moreover numerical integration of the ordinary differential equations (odes) presented in section 2 are performed through the variable order solver ode15s [31] available within the mathematical software package Matlab. The normalized time interval goes from 0 to 50000 in all simulations. As explained in section 3, taking into account that ode15s is a variable step ode solver, for each simulation the synchronization threshold for Eq. (19) was set to two times the minimum integration time step required by the solver. This is a very severe condition for synchronization, which is hard to apply in neural networks. However this study does not intend to reproduce quantitatively some biological phenomenon. It rather aims at gaining a better understanding of the key role of memristor dynamics in the emergence of synchronous behavior in a very simple bio-inspired network.

The following remark defines the methodology adopted for studying the synchronization properties of the DN for a fair comparison with the dynamics of the MN.

Remark 1 With regards to the MN, in case the two neurons act in unison, since $x_1(t) = x_2(t)$, from Eq. (8) we have $\dot{x} = 0$ and $x$ converges to an equilibrium point $x_{\text{ss}}$, located either within or outside the central portion of the charge-flux relationship given in Eq. (7). Using Eq. (9), in the first (latter) case the steady state dimensionless normalized memductance (let us denote it as $W_{\text{ss}} = W(x_{\text{ss}})$) attains the lower (upper) value $W_{\text{ss},l} = \hat{\alpha}$ ($W_{\text{ss},u} = \hat{\beta}$). Therefore, for each value of $\varepsilon$, the steady state effective coupling strength (here indicated as $C_{\text{ss}} = C(x_{\text{ss}})$) may assume two values: a lower one $C_{\text{ss},l} = \varepsilon W_{\text{ss},l} = \varepsilon \hat{\alpha}$ if $x_{\text{ss}} < \hat{\phi}_c$, and an upper one $C_{\text{ss},u} = \varepsilon W_{\text{ss},u} = \varepsilon \hat{\beta}$ if $x_{\text{ss}} > \hat{\phi}_c$. Recalling that $\varepsilon$ is varied in $[\varepsilon_1, \varepsilon_f] = [10^{-5}, 10^{-3}]$ with step $\Delta \varepsilon = 5 \cdot 10^{-5}$, possible values for $C_{\text{ss},l}$ are distributed
in \( \hat{a}[\varepsilon_i, \varepsilon_f] = [10^{-4}, 10^{-2}] \) and separated by a distance of \( \hat{a}\Delta \varepsilon = 5 \cdot 10^{-4} \). Similarly, \( C_{ss,u} \) may only assume values located in \( \hat{\beta}[\varepsilon_i, \varepsilon_f] = [10^{-2}, 1] \) with spacing \( \hat{\beta}\Delta \varepsilon = 5 \cdot 10^{-2} \). As a result, it is sufficient to interrogate the synchronization capability of the DN by stepping the coupling strength \( C \) through all the above mentioned possible values of \( C_{ss,l} \) and of \( C_{ss,u} \).

This remark clarifies what it is meant by “equivalent diffusive and memristor networks” throughout this paper. Following this remark, we checked whether or not synchronization emerges in a DN with coupling strength \( C \) varied in \([0, 10^{-2}]\) with step \( \Delta C = 10^{-4} \leq 5 \cdot 10^{-4} \) and then in \([10^{-2}, 1]\) with step \( \Delta C = 10^{-2} \leq 5 \cdot 10^{-2} \). Table I reports for each \( I \) value of interest the synchronization windows of the DN within the covered interval \([0, 1]\) of the coupling strength together with the type of behavior under exam.

<table>
<thead>
<tr>
<th>( I )</th>
<th>Behavior</th>
<th>Synchronization windows within the ( C ) interval ([0, 1])</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.4</td>
<td>one-spike-per-burst bursting</td>
<td>([0.6 \cdot 10^{-3}, 27 \cdot 10^{-3}] \cup [0.159, 1])</td>
</tr>
<tr>
<td>1.8</td>
<td>two-spike-per-burst bursting</td>
<td>([0.6 \cdot 10^{-3}, 15 \cdot 10^{-3}] \cup [0.343, 0.376, 1])</td>
</tr>
<tr>
<td>3</td>
<td>chaos</td>
<td>([0.427, 0.428, 0.430, 1])</td>
</tr>
<tr>
<td>3.45</td>
<td>period-one spiking</td>
<td>([0.2 \cdot 10^{-3}, 0.9 \cdot 10^{-3}] \cup [2.9 \cdot 10^{-3}, 3.2 \cdot 10^{-3}] \cup [0.522, 1])</td>
</tr>
<tr>
<td>4</td>
<td>period-two spiking</td>
<td>([0.1 \cdot 10^{-3}, 3.9 \cdot 10^{-3}] \cup [0.521, 1])</td>
</tr>
</tbody>
</table>

These results do not match those reported in [29], where some of the given synchronization windows were unnoticed. The following section reports the results of the study of the synchronization properties of the MN for some value of the input \( I \) to the two identical HR cells.

4.1 Simulation results

Let us now turn our attention on the behavior of the MN. In each case where synchronized behavior shall be observed, we shall also check what takes place in an equivalent DN with coupling resistance equal to the steady state value of the memristance of the synapse in the MN. The values for the parameters of the memristor model are taken as specified in section 4.

4.1.1 Case \( I = 1.4 \)

Let us first choose \( I = 1.4 \). Figures 1(a1)–(a3) refer to the behavior of the memristor network for a coupling strength coefficient \( \varepsilon = 21 \cdot 10^{-6} \) and a memristor initial state \( x(0) = 0 \). In particular, Fig. 1(a1) reports the time waveforms of \( x_1(t) \) (in black) and \( x_2(t) \) (in red), which lock into synchronous one-spike-per-burst bursting behavior. After discarding the transitory dynamics, peaks in the two signals occur at time instants satisfying the synchronization condition defined by Eq. (19). Figure 1(a2) shows one pair of spikes in \( x_1 \) and \( x_2 \). In Fig. 1(a3) \( x_1 \) is plotted versus \( x_2 \). As expected, a straight line is drawn.

Figures 2(a)–(b) display the time behavior of the memristor state and of the dimensionless normalized memductance respectively. Since the memristor state approaches an equilibrium lying within the central region of the charge-flux characteristic expressed by Eq. (7), \( \dot{W}(x) \) assumes the lower value \( \dot{W}_{ss,l} = \dot{a} = 10 \) at steady state. As a result, for the given value for \( \varepsilon \), at the end of transients the effective coupling strength settles to the lower value \( C_{ss,l} = \varepsilon \dot{W}_{ss,l} = \varepsilon \dot{a} = 2.1 \cdot 10^{-4} \). This value is below the first synchronization window of the DN for \( I = 1.4 \) (see Table I). In fact, setting the coupling strength \( C \) of the DN to \( 2.1 \cdot 10^{-4} \), asynchronous dynamics develop in such network. The two neuron oscillators undergo the expected one-spike-per-burst bursting behavior (see Fig. 1(b1)). However, they do not oscillate in synchrony. There is a misalignment between adjacent peaks, as it is demonstrated in Fig. 1(b2), and the synchronization condition given in Eq. (19) does not hold. The projection of the attractor of the system onto the \( x_1-x_2 \) plane, visualized in Fig. 1(b3), is yet another signature for the asynchronism between the dynamics of the diffusively coupled oscillators.

Let us now keep the value of \( I \) unaltered and set \( \varepsilon = 1.55 \cdot 10^{-4} \) and \( x(0) = 0.5 \) in the MN. Once again synchronous one-spike-per-bursting behavior emerges in the network (see Figs. 3(a1)–
Fig. 1. One-spike-per-burst bursting behavior for $I = 1.4$. Plots (a1)–(a3): synchronous dynamics in the MN for $\varepsilon = 21 \cdot 10^{-6}$ and $x(0) = 0$. Plots (b1)–(b3): asynchronous dynamics in the equivalent DN with $C = 2.1 \cdot 10^{-4}$.

Fig. 2. Time waveforms of memristor state $x$ and dimensionless normalized memductance $\hat{W}(x)$ for the same simulation setting as in Figs. 1(a1)–(a3). At steady state $\hat{W}(x)$ sets to $W_{ss,l} = 10$, and, for $\varepsilon = 21 \cdot 10^{-6}$, $C(x)$ settles at $C_{ss,l} = 2.1 \cdot 10^{-4}$.

Fig. 3. Synchronous (plots (a1)–(a3)) and asynchronous (plots (b1)–(b3)) one-spike-per-burst bursting behaviors occurring respectively in a MN for $\varepsilon = 1.55 \cdot 10^{-4}$ and $x(0) = 0.5$ and in an equivalent DN with $C = 0.155$. As in the previous case study $I$ is kept equal to 1.4.

(a3), where the three plots respectively depict the same signals as in Figs. 1(a1)–(a3)). In this occasion the memristor state converges to an equilibrium lying outside the central region of the PWL $q$-$\varphi$ characteristic of Eq. (7) (see Fig. 4(a)), and, consequently, the dimensionless normalized memductance settles to the upper value $\hat{W}_{ss,u} = \hat{\beta} = 1.03$ at steady state (see Fig. 4(b)). It follows that, for the given value of $\varepsilon$, the effective coupling strength assumes the upper value $C_{ss,u} = \varepsilon \hat{\beta} =$
This value lies between first and second synchronization window of the DN for $I = 1.4$ (see Table I). Figures 3(b1)–(b3), visualizing the same signals as in Figs. 1(b1)–(b3) respectively, confirm the asynchronism between the one-spike-per-burst bursting behaviors of the two HR oscillators of the equivalent DN with $C = 0.155$.

In the following we shall investigate how the synchronization properties of the MN change as two control parameters, namely $\varepsilon$ and $x(0)$, are varied. Stepping $\varepsilon$ in $[\varepsilon_i, \varepsilon_f] = [10^{-5}, 9.6 \cdot 10^{-4}]$ with spacing $\Delta \varepsilon = 5 \cdot 10^{-5}$ (a total of 20 points) and $x(0)$ in $[-1.5, +1.5]$ with separation $\Delta x(0) = 0.25$ (a total of 13 points), we recorded the synchronization properties of the MN for each $(x(0), \varepsilon)$ pair.

As shown in the pseudo-color checkerboard plot of Fig. 5, within the previously specified simulation time the MN network is unable to synchronize in some cases (blue color-filled square cells), while exhibits synchronous one spike per burst bursting behavior in all other scenarios (light-blue, ochre, and magenta color-filled square cells).

For all cases where the MN cells were found to oscillate in synchrony, the steady state effective control parameter $C_{ss}$ was recorded. Each light-blue (ochre)-filled cell identifies a pair $(x(0), \varepsilon)$ for which $C_{ss}$ falls into first (second) synchronization window of the DN for $I = 1.4$ (see Table I). On the other hand, the magenta-filled cell identifies pair $(x(0), \varepsilon) = (-0.25, 10^{-5})$ for which $C_{ss}$ falls somewhere outside the two synchronization windows of the DN for $I = 1.4$. In particular, at the end of transients $\hat{W}(x)$ assumes the lower value $\hat{W}_{ss} = \hat{\alpha}$. Therefore the steady state value for $C(x)$ is $C_{ss,t} = \varepsilon \hat{\alpha} = 0.1 \cdot 10^{-3}$, which lies below the first synchronization window of the DN for $I = 1.4$ (see Table I).

Note that yet another information may be drawn from the checkerboard plot of Fig. 5. In case a cross (diamond) marker is present in the center of a cell corresponding to a scenario where the MN locks into synchronous behavior, then at steady state $\hat{W}(x)$ settles to its lower (upper) value $\hat{W}_{ss,l} = \hat{\alpha}$ ($\hat{W}_{ss,u} = \hat{\beta}$).

A high-resolution study of the synchronization properties of the MN for $I = 1.4$ was also carried out. Parameter $\varepsilon$ was stepped from $\varepsilon_i = 10^{-6}$ to $\varepsilon_f = 10^{-3}$ with spacing $\Delta \varepsilon = 5 \cdot 10^{-6}$. In line with the observations outlined in the remark 1, here, in case the MN locks into synchronization, $C_{ss,t}$ may assume values lying in closed interval $[10^{-5}, 10^{-2}]$ and separated by $5 \cdot 10^{-5}$, while the possible values of $C_{ss,u}$ are distributed in $[10^{-3}, 1]$ with spacing $5 \cdot 10^{-3}$. In this high-resolution study of the $I = 1.4$ case, the synchronization windows of the DN reported in Table I need to be merged through a union set operation with the synchronization intervals found by stepping $C$ from 0 to $10^{-2}$ with separation $10^{-5} \leq 5 \cdot 10^{-5}$. The updated synchronization windows of the DN for $I = 1.4$ are expressed by:

$$C \in [5.1 \cdot 10^{-4}, 2.7 \cdot 10^{-2}] \cup [0.159, 1].$$

(20)

Varying $\varepsilon$ in $[\varepsilon_i, \varepsilon_f] = [10^{-6}, 9.96 \cdot 10^{-4}]$ with step size $\Delta \varepsilon = 5 \cdot 10^{-6}$ (a total of 200 points) and $x(0)$ in $[-1.5, +1.5]$ with step size $\Delta x(0) = 0.25$ (a total of 13 points), the number of magenta-filled cells, identifying pairs $(x(0), \varepsilon)$ for which $C_{ss}$ falls somewhere outside the two synchronization windows of
Asynchronous (blue-filled cells) and synchronous (light-blue, ochre, and magenta-filled cells) one-spike-per-burst bursting oscillatory behavior in the MN. For the \((x(0), \varepsilon) = (-0.25, 10^{-5})\) pair identifying the magenta-filled cell the equivalent DN was unable to lock into synchronous dynamics. The light-blue (ochre)-filled cells refer to \((x(0), \varepsilon)\) pairs for which \(C_{ss}\) falls into first (second) synchronization window of the DN. In case the MN neurons oscillate in unison, if the dimensionless normalized memductance settles to \(\dot{\alpha} (\dot{\beta})\) a cross (diamond) marker appears in the middle of the corresponding checkerboard cell. The input is set to \(I = 1.4\).

High-resolution study of the \(I = 1.4\) case. Checkerboard plot in the \(x(0)-\varepsilon\) plane. MN out of synchrony (blue-filled cells). MN in synchrony and DN out of synchrony (magenta-filled cells). MN and DN in synchrony (all remaining cells). In the light-blue (ochre)-filled cells \(C_{ss}\) falls into first (second) synchronization window of the DN (see Eq. (20)). The significance of the markers is explained in the text.

the DN (see Eq. (20)), increases, as shown in the pattern of Fig. 6, where the blue-filled cells refer to cases where the memristor-coupled neurons oscillate asynchronously, while in all other cases both the MN and the equivalent DN lock into synchronous behavior with \(W_{ss}\) equal to its lower (upper) value in case a cross (diamond) marker is placed inside the checkerboard cell. In particular, light-blue- and ochre-filled cells respectively refer to cases where \(C_{ss}\) falls into first and second synchronization window of the DN. Note that in Fig. 6 only a portion of the covered interval of the coupling strength coefficient \(\varepsilon\) is visualized, i.e. from \(10^{-6}\) to \(1.76 \cdot 10^{-4}\) (a total of 36 points).

Within the checkerboard plot of Fig. 6, only in the case of pair \((x(0), \varepsilon) = (1.56 \cdot 10^{-4}, -0.25)\) (top-most magenta-filled cell) after transients are elapsed does \(W(x)\) approach \(W_{ss,u} = \dot{\beta} = 10^3\) and therefore does \(C(x)\) attain a value falling between the two synchronization windows of the equivalent
DN (see Eq. (20)), i.e. $C_{ss,u} = \varepsilon \hat{W}_{ss,u} = \varepsilon \hat{\beta} = 0.156$. Note, however, that this is not an isolated case: Figs. 3–4 show another example where $\hat{W}(x_{ss}) = W_{ss,u}$ and $C(x_{ss})$ falls between the two synchronization windows of the DN.

Anyways, within the checkerboard plot of Fig. 6, for all other magenta-filled cells the steady state value for the dimensionless normalized memductance was found to be $\hat{W}_{ss,l} = \hat{\alpha} = 10$, and, consequently, after transients the effective coupling strength fell somewhere below the first synchronization window of the DN (see Eq. (20)). The simulation results of Figs. 1–2 refer to the bottom-most magenta cell in Fig. 6.

### 4.1.2 Case $I = 1.8$

Let us now investigate the dynamical behavior of the MN in case each of the two identical HR neurons are driven with $I = 1.8$. For such excitation the uncoupled HR neuron experiences two-spike-per-burst bursting behavior. Figures 7(a1)–(a3), graphically reporting the same signals as Figs. 1(a1)–(a3) respectively, show that the two-spike-per-burst bursting behaviors of the memristor-coupled HR neurons occur in unison for $\varepsilon = 3.1 \cdot 10^{-5}$ and $x(0) = 0.75$. Here $x$ converges to a value located within the central region of the PWL $q$-$\varphi$ nonlinearity of Eq. (7) (see Fig. 8(a)). Thus the steady state dimensionless normalized memductance is $\hat{W}_{ss,l} = \hat{\alpha} = 10$ (see Fig. 8(b)) and the effective coupling strength attains the value of $C_{ss,l} = \varepsilon \hat{W}_{ss,l} = 3.1 \cdot 10^{-4}$ after transients are elapsed. This value is below the first synchronization window of the DN for $I = 1.8$ (see Table I). Therefore, as confirmed in Figs. 7(b1)–(b3), where the same signals as in Figs. 1(b1)–(b3) are shown respectively, the equivalent DN with coupling strength $C = C_{ss,l} = 3.1 \cdot 10^{-4}$ fails to lock into synchronization.

![Fig. 7](image)

**Fig. 7.** Synchronous and asynchronous two spike-per-burst bursting behaviors of the MN for $\varepsilon = 3.1 \cdot 10^{-5}$ and $x(0) = 0.75$ (plots (a1)–(a3)) and for the equivalent DN with $C = C(x_{ss}) = 3.1 \cdot 10^{-4}$ (plots (b1)–(b3)). The input to the identical oscillators is set to $I = 1.8$.

![Fig. 8](image)

**Fig. 8.** Time waveforms of $x$ (plot (a)) and $\hat{W}(x)$ (plot (b)) for the simulation scenario of Figs. 7(a1)–(a3). Note that $\hat{W}(x_{ss}) = \hat{W}_{ss,l} = \hat{\alpha} = 10$. For $\varepsilon = 3.1 \cdot 10^{-5}$ this leads to $C(x_{ss}) = C_{ss,l} = \varepsilon \hat{\alpha} = 3.1 \cdot 10^{-4}$. 

302
Fig. 9. Plots (a1)–(a3): synchrony between the two spike-per-burst bursting behaviors of the MN cells for $\varepsilon = 36 \cdot 10^{-5}$ and $x(0) = 0.5$. Plots (b1)–(b3): asynchrony between the three spike-per-burst bursting behaviors of the DN cells for $C = C(x_{ss}) = 0.36$.

Fig. 10. Time evolution of $x$ and $\hat{W}(x)$ under the simulation setting in Figs. 9(a1)–a(3). Note that after transients $\hat{W}(x)$ settles to $\hat{W}_{ss,u} = 10^{8}$. For $\varepsilon = 36 \cdot 10^{-5}$, this leads to $C(x_{ss}) = C_{ss,u} = 0.36$.

Even more interesting is the following simulation result. Letting $\varepsilon = 36 \cdot 10^{-5}$ and $x(0) = 0.5$ the MN cells oscillate synchronously on the expected behavior (see Figs. 9(a1)–(a3)). In this case the memristor state $x$ converges to an equilibrium located off the central portion of the PWL charge-flux relationship of Eq. (7) (see Fig. 10(a)) and, consequently, $\hat{W}(x)$ settles to its upper value $\hat{W}_{ss,u} = 10^{3}$ at steady state (see Fig. 10(b)). This leads to a steady state effective coupling strength $C(x_{ss}) = C_{ss,u} = \varepsilon \hat{W}_{ss,u} = 0.36$, falling between the isolated synchronization value $C = 0.343$ and the second synchronization window of the DN for $I = 1.8$ (see Table I). It follows that the equivalent DN with $C = C(x_{ss}) = 0.36$ is unable to exhibit synchronization. Further, as shown in Figs. 9(b1)–(b3), the two asynchronous diffusively coupled neurons undergo a dynamical behavior differing from the expected one, particularly three-spike-per-burst bursting behavior.

Figure 11 depicts the checkerboard plot in the plane $x(0)$–$\varepsilon$ for $I = 1.8$. Here the initial condition on the memristor state is varied from $-1.5$ to $1.5$ with spacing $\Delta x(0) = 0.25$ and $\varepsilon$ is stepped in $[10^{-5}, 9.6 \cdot 10^{-4}]$ with separation $\Delta \varepsilon = 5 \cdot 10^{-5}$. Black-filled cells denote cases where the MN fails to lock into synchronous behavior, while in all other scenarios the MN neurons oscillate in unison. For each light-blue (ochre)-filled cell $C_{ss}$ falls into first (second) synchronization window of the DN (see Table I). The magenta-filled cells refer to situations where the dynamics of MN neurons are synchronous, while the equivalent DN is unable to exhibit synchronous dynamics. Note that no cell identifies a pair leading to an effective coupling strength equal to the isolated synchronization value $C = 0.343$ in Table I.

Almost a full row of magenta-filled cells at $\varepsilon = 36 \cdot 10^{-5}$ (only the cell at $x(0) = 1.25$ makes exception) appear in Fig. 11. Each of these cells identifies a pair $(x(0), \varepsilon)$ leading to $\hat{W}(x_{ss}) = \hat{W}_{ss,u} = \beta = 10^{3}$.  

303
This implies $C(x_{ss}) = C_{ss,u} = \varepsilon \beta = 0.36$, which falls between the isolated synchronization value $C = 0.343$ and the second synchronization window of the DN for $I = 1.8$ (see Table I). Note that the dynamics of the system for the magenta-filled cell at $(x(0), \varepsilon) = (0.5, 36 \cdot 10^{-5})$ (the only cell of the checkerboard of Fig. 11 filled with magenta and located below an adjacent light-blue cell) were previously described (see Figs. 9 and 10).

A cross or diamond marker is inserted in the center of each cell where the MN exhibits synchronous behavior. The first and latter markers respectively denote cases where the dimensionless normalized memductance settles to $\hat{\alpha}$ and $\hat{\beta}$.

The same high-resolution study conducted for the case $I = 1.4$ was carried out for the case $I = 1.8$. Parameter $\varepsilon$ was varied from $\varepsilon_i = 10^{-6}$ to $\varepsilon_f = 10^{-3}$ with spacing $\Delta \varepsilon = 5 \cdot 10^{-6}$ ($x(0)$ was stepped from $-1.5$ to $1.5$ with separation 0.25). Following the procedure described for the high-resolution study of the case $I = 1.4$, the result of the investigation of the synchronization properties of the equivalent DN for $I = 1.8$ (see Table I) was properly merged in a set theoretic union operation with the results of the analysis of the dynamics of the diffusively-coupled neurons under variation of the coupling strength $C$ in $[0, 10^{-2}]$ with step $\Delta C = 10^{-5}$. The combination of these results led to the following synchronization windows of the DN for the high-resolution study of the case $I = 1.8$:

$$C \in 6 \cdot 10^{-4} \cup 6.6 \cdot 10^{-4} \cup [6.8 \cdot 10^{-4}, 7.3 \cdot 10^{-4}] \cup [7.5 \cdot 10^{-4}, 15 \cdot 10^{-3}] \cup 0.343 \cup [0.376, 1]. \quad (21)$$

In the following discussion we shall number the 3 isolated synchronization values and the 3 synchronization windows in increasing order. In Fig. 12 only a portion of the spanned range of $\varepsilon$ is shown, particularly the 80 points from $10^{-6}$ up till $396 \cdot 10^{-6}$ with spacing $5 \cdot 10^{-6}$. The color coding scheme is the following. The black-filled cells denote an out-of-synchrony MN. In the cases where the MN is in synchrony while the equivalent DN is out of synchrony, the relevant cell is filled with magenta. In all other scenarios both the MN and the DN are locked into the expected synchronous behavior with $C(x_{ss})$ lying within the first synchronization window (light-blue-filled cells), within the second synchronization window (ochre-filled cells), and on the second isolated synchronization value (white-filled cells). In this checkerboard the MN does never lock into synchronization with $C_{ss}$ equal to the first or third isolated synchronization value in Eq. (21). The cross and diamond markers inside the cells referring to synchronous dynamics in the MN indicate that the dimensionless normalized memductance assumes after transients the lower and upper value respectively.

Let us gain some more insight into the dynamics of the system for the magenta-filled cells, which add up to 100 in the checkerboard of Fig. 12. Let us examine where does the the effective coupling strength $C(x)$ falls at steady state with respect to the synchronization windows of the equivalent DN, reported in (20). Starting off from the bottom of the checkerboard, it is worthy to notice that $C(x_{ss})$
Fig. 12. Synchronization properties of the MN for the high-resolution study of the $I = 1.8$ case. See text for details on the color coding scheme.
falls between second synchronization window and third isolated synchronization value for the lowest 6 cells with a diamond marker. Continuing towards the top of the checkerboard, we find 18 cells with a cross marker. Of these cells, the lowest 12 are characterized by a steady state effective coupling strength lying below the first isolated synchronization value, while the upper 6 cells located at row \( \varepsilon = 61 \cdot 10^{-6} \) exhibit synchronous behavior for a value of \( C(x_{ss}) \) between first and second isolated synchronization value. Finally, for the top-most 76 magenta-filled cells with diamond marker \( C(x_{ss}) \) is larger than the third isolated synchronization value but lower than the lower edge of the third synchronization window.

Note that the behavior of the system for the cell filled with magenta and identified by a memristor initial state equal to 0.75 and by a coupling strength coefficient \( \varepsilon \) of value \( 3.1 \cdot 10^{-5} \) (the lowest magenta-filled cell located in the column identified by the given value for \( x(0) \) in the checkerboard of Fig. 12) was described in Figs. 7 and 8.

In conclusion, the simulations described in sections 4.1.1 and 4.1.2 have given some insight into the dynamics of a simple network of two memristor-coupled HR neurons. The study of the synchronization properties of the MN for other values of the input \( I \) would not add much to the main result of our investigations, i.e. the crucial role that the time history of the state of the memristor synapse may have on the mechanisms behind the emergence of synchronization in the dynamical system under exam. We prefer to direct our efforts towards the development of some analytical methodology which may allow one to understand why in some cases synchronization only develops in the MN, whereas the equivalent DN exhibits asynchronous behavior (magenta-filled cells in the four checkerboards of Figs. 5, 6, 11, and 12).

5. Conclusions

The memristor manifests qualities and behaviors which are naturally observed in biological synapses [32, 33]. It exists at the nano-scale, it consumes very little power, it is ideally suited for parallel processing, it may process and store information simultaneously, and, last but not least, it offers a conductive behavior depending on the time evolution of the flux through it. Furthermore it is able to reproduce rules governing the process of neural learning, including the Hebbian rule, Spike-Time-Dependent-Plasticity and Spike-Rate-Dependent-Plasticity [17]. It is thus timely to investigate the dynamics of novel neuromorphic networks where memristors replace conventional electronic implementations of biological synapses.

This work analyzes the mechanisms underlying the synchronization phenomena occurring in a very simple bio-inspired network of Hindmarsh-Rose neurons coupled through an ideal flux-controlled memristor. The influence of the memory capability of this synaptic coupling on the emergence of synchronization is demonstrated through a comparison between the synchronization dynamics developing in the Memristor Network and the behaviors observed in an equivalent Diffusive Network, where, for each synchronization scenario in the MN, the value of the resistive synapse is taken equal to the steady state memristance of the memristor.

The results of this investigation give further insight into the peculiar behavior of the memristor, which has been fascinating the research community since the publication of Chua’ seminal paper [6] and has attracted the attention of scientists interested in neuromorphic applications for six years, when Hewlett Packard Fellow Stan Williams gave the first conscious experimental evidence for the natural existence of memristance at the nano-scale [7].

References


