Paper

Nonlinear control by coupled oscillator: from nonholonomic systems to quasi-passive dynamic walker

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Received January 24, 2015; Revised May 1, 2015; Published October 1, 2015

Abstract: In this paper, we propose a novel control method for nonholonomic constrained systems using coupled oscillators composed through the Kuramoto model. The Kuramoto model is a mathematical model used to describe synchronization phenomena. We focus on the Kuramoto model because it is able to generate stable rhythmic signals that are modulable for engineering, and because it is sufficiently simple for mathematical analysis. The contributions of this paper are twofold. First, we propose the use of a pair of Kuramoto oscillators as dynamic controllers for a nonholonomic Brockett integrator system; next, we define a feedback control scheme by adjusting its angular velocities based on the system state. We then derive stability criteria for the entire feedback system and examine the effectiveness of the system through several numerical simulations. We also show that the proposed idea is applicable to a two-wheeled vehicle system, which is locally equivalent to the Brockett integrator under proper coordinate and input transformations. Second, we apply the proposed method to control the walking distance of a quadrupedal quasi-passive dynamic walker and examine the effectiveness of the proposed method through walking experiments.

Key Words: nonholonomic constrained system, kuramoto oscillator model, Brockett integrator, two-wheeled vehicle, quasi-passive dynamic walking

1. Introduction

In the field of nonlinear control theory, nonholonomic constrained systems have been extensively investigated since the early 90s [1]. Nonholonomic systems are literally defined as a class of mechanical that undergo nonintegrable mechanical constraints. From the viewpoint of nonlinear control, one outstanding feature is that typical nonholonomic systems are not controllable in the linear sense (i.e., corresponding linear approximations fail to be controllable), but they can be controllable in a nonlinear sense by considering the effects of Lie brackets on the input vector fields. Such systems

Fig. 1. Relationship among Kuramoto oscillator model, Brockett integrator, Two-wheeled vehicle, Quadrupedal Quasi-Passive Dynamic Walker and the proposed control method in this paper.

often appear in locomotion control problems of mobile or space robots.

A key control strategy for nonholonomic systems involves the use of *periodic control inputs with proper phase gaps*, such as sine-cosine pairs, because the gaps are approximately proportional to the Lie bracket effects. In this case, the problem is how to create periodic inputs with a proper phase gap. In our previous work [2], a Matsuoka model, which is a mathematical model of neural elements proposed by Matsuoka [3], was used to generate such periodic control inputs. The proposed control strategy was applied to a rolling sphere control problem, and its effect was verified with numerical simulations. However, a sufficient mathematical stability analysis of the proposed method has not yet been conducted.

In this paper, we propose a novel control method for nonholonomic constrained systems based on the Kuramoto model, which is a mathematical model used to describe synchronization phenomena [4–6]. We focus on the Kuramoto model because it is able to generate stable rhythmic signals that are modulable through engineering, and because it is sufficiently simple for mathematical analysis.

Our primary contribution is roughly composed of two parts. First, we propose the use of a pair of Kuramoto oscillators as dynamic controllers for a nonholonomic Brockett integrator system, and then define a feedback control scheme by adjusting its angular velocities based on the system state. We then derive stability criteria for the entire feedback system and examine its effectiveness through several numerical simulations. We also show that the proposed idea is applicable to a two-wheeled vehicle system, which is locally equivalent to a Brockett integrator under proper coordinates and input transformations. Next, we apply the proposed method to a completely different target: a quadrupedal quasi-passive dynamic walking robot. This walking robot, which we called “Duke,” was developed in our previous study [7]. In fact, we revealed that Duke’s walking dynamics reflect *nonholonomic behavior* due to the rolling contact constraints between its feet and the ground, and that Duke is partially similar to a two-wheeled vehicle. This result motivated us to connect these two apparently unrelated topics: the Kuramoto model and the walking robot.

This paper is organized as follows (Fig. 1). Section 2 introduces the Brockett integrator and explains its basic features. Section 3 introduces the Kuramoto oscillator model and describes its interesting properties. In Section 4.1, we propose a novel control method using the Kuramoto model for a nonholonomic constraint system, discuss the stability of the proposed control method, and examine its effectiveness with several numerical simulations. In Section 4.2, the proposed control method is applied to a two-wheeled vehicle system. Section 5 demonstrates the application of the proposed control method to a quadrupedal quasi-passive-dynamic-walking robot and shows the experimental results. Conclusions and future works are discussed in Section 6.

2. Brockett integrator

First, we focus on the so-called Brockett integrator system [8] in particular, which is the simplest example of a nonholonomic system. The state equation of the Brockett integrator is shown as follows:

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ X_2 & -X_1 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \quad (1)$$

$$:= \mathbf{g}_1(\mathbf{X})U_1 + \mathbf{g}_2(\mathbf{X})U_2 \quad (2)$$

where $\mathbf{X} = (X_1, X_2, X_3)^T \in \mathbb{R}^3$ is the system state and $\mathbf{U} = (U_1, U_2)^T \in \mathbb{R}^2$ is the control input; $\mathbf{g}_1$ and $\mathbf{g}_2$ are the input vector-fields corresponding to the inputs U_1 and U_2 respectively. It is clear that X_1 and X_2 can be controlled directly by U_1 and U_2 , whereas X_3 should be controlled indirectly by appropriate coordination of $U_1(t)$ and $U_2(t)$, considering the time profiles. For instance, it was shown that periodic inputs U_1 and U_2 with proper phase gaps, such as sine-cosine pairs of functions [1] were effective to generate net change for X_3 .

Here, let us define a nonassociative, antisymmetric product between two smooth vector-fields, $\mathbf{g}_i$ and $\mathbf{g}_j$, called *Lie bracket*:

$$[\mathbf{g}_i, \mathbf{g}_j] = \frac{\partial \mathbf{g}_j}{\partial \mathbf{X}} \mathbf{g}_i - \frac{\partial \mathbf{g}_i}{\partial \mathbf{X}} \mathbf{g}_j. \quad (3)$$

Lie bracket plays a significant role in controlling nonlinear systems of the form (2), since it gives us an approximate change of the state vector X under periodic control inputs. Suppose $U_1(t)$ and $U_2(t)$ are functions of t with period T and zero-average (thus $\int U_1(t)dt$ and $\int U_2(t)dt$ are also T -periodic). The change of X after one cycle of periodic control, say $X(T) - X(0)$, is approximately parallel to the vector $[g_1, g_2]$. Moreover, the amount of this displacement is proportional to the area enclosed by the curve $(\int U_1(t)dt, \int U_2(t)dt)$. This is derived from a differential-geometric principle, called area formula [1, 9], which is essentially a generalization of the Stokes' theorem.

For example, the Lie bracket of $\mathbf{g}_1$ and $\mathbf{g}_2$ in the system (1) can be calculated as follows:

$$[\mathbf{g}_1, \mathbf{g}_2] = [0, 0, -2]^T. \quad (4)$$

The third element of this Lie bracket is nonzero, thus, system (1) can also be maneuvered in the X_3 direction.

From these facts, it is expected that X_3 can be controlled with certain periodic U_1 and U_2 . In this case, the problem is how to create periodic inputs with a proper phase gap. This is because the area enclosed by the curve of input integrations strongly depends on a phase gap between U_1 and U_2 . Actually, when $u_1 = A \sin t$ and $u_2 = A \sin(t - \psi)$ are used as periodic inputs, the enclosed area S , by the curve of input integrations becomes

$$S = -A^2 \sin \psi. \quad (5)$$

In the next section, we propose the use of a couple of Kuramoto oscillators to produce appropriate phase gaps by modulating their parameters.

Remark 1. *The Brockett integrator (1) is known as a canonical representation of nilpotent nonholonomic systems. Other equivalent representations include chained systems [10] and power forms [11]. This class of systems can be generalized as “first-order systems [12]”, where $\mathbf{g}_1(X), \mathbf{g}_2(X)$ and $[\mathbf{g}_1, \mathbf{g}_2](X)$ are linearly independent to each other at any X . Since the proposed method only based on the aforementioned area formula, the method would be applied to the first-order systems with slight customization.*

3. Kuramoto oscillator model

It is well known that there are many types of synchronization phenomena in autonomous systems, such as mechanical systems, chemical reactions, and circadian rhythms. To describe such synchronization,

Fig. 2. Simulation results of Kuramoto oscillator model.

various models have been proposed. The Kuramoto model, first proposed by Kuramoto [4], is a mathematical model used to describe synchronization [4, 5]. The Kuramoto model, which consists of a population of N coupled phase oscillators θ_i having natural frequencies ω_i and coupling strength k_{ij} , is described as

$$\frac{d\theta_i}{dt} = \omega_i + \sum_{j=1}^N \frac{k_{ij}}{N} \sin(\theta_j - \theta_i). \quad (6)$$

This model is simple enough to be mathematically tractable, yet sufficiently complex to be non-trivial. For example, when k_{ij} is sufficiently weak, the oscillators run incoherently, whereas beyond a certain threshold, collective synchronization spontaneously emerges. Acebron et al. studied many different models, such as nearest-neighbor coupling, hierarchical coupling, random long-range coupling, or even state-dependent interactions [6]. Moreover, the behavior of each oscillator varies depending on ω_i . Figure 2 shows numerical simulation results of the Kuramoto model, which consists of two oscillators θ_1, θ_2 :

$$\frac{d\theta_1}{dt} = \omega_1 + k_{12} \sin(\theta_2 - \theta_1), \quad \frac{d\theta_2}{dt} = \omega_2 + k_{21} \sin(\theta_1 - \theta_2). \quad (7)$$

Figures 2(a) and (b) show simulation results for when natural frequencies were set as $\omega_1 = \omega_2 = \pi$ and $\omega_1 = 1.05\pi, \omega_2 = 0.95\pi$, respectively. Other parameters were $k_{12} = k_{21} = 1$ and initial conditions were $\theta_1(0) = 1$ and $\theta_2(0) = \pi$. The two oscillators in Fig. 2(a) are synchronized. Indeed $\sin \theta_1 - \sin \theta_2$ converges to zero since $\frac{d}{dt}(\theta_1 - \theta_2) = -2\sin(\theta_1 - \theta_2)$. On the other hand, those in Fig. 2(b) are decoupled. This means that a phase gap between two oscillators can be produced through even a small difference in ω_i .

Thus, in this study, we use this dependency on the natural frequency of the Kuramoto model to produce periodic control inputs for the Brockett integrator.

4. Feedback control for nonholonomic systems using the Kuramoto model

4.1 Control of Brockett integrator

In this section, we propose a novel control method for nonholonomic constrained systems using coupled oscillators composed by using the Kuramoto model. Next, we derive the stability criteria of the Brockett integrator with the proposed method. Finally, we examine the effectiveness of the proposed method through several numerical simulations

4.1.1 Proposed control method

Based on Kuramoto model (7), let us consider the following oscillator model with a feedback term,

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} = \begin{pmatrix} (1 + K_p X_3) \omega_0 + k \sin(\theta_2 - \theta_1) \\ (1 - K_p X_3) \omega_0 + k \sin(\theta_1 - \theta_2) \end{pmatrix}, \quad (8)$$

where ω_0 is the common natural frequency, k is the bonding strength of oscillators, and K_p is the feedback gain. As shown, oscillator model (8) includes state X_3 , which modifies the frequency of each oscillator. Consequently, it is expected that these oscillators will show various synchronization patterns depending on state $\mathbf{X}$.

Using these oscillators, θ_1 and θ_2 , we propose to assign the following U_1 and U_2 as the control inputs for Brockett integrator (1),

$$U_1 = A \cos(\theta_1) \dot{\theta}_1, \quad U_2 = B \cos(\theta_2) \dot{\theta}_2, \quad (9)$$

where $\int U_1 dt = A \sin \theta_1$ and $\int U_2 dt = B \sin \theta_2$.

4.1.2 Stability analysis

In this section, we show that the proposed control inputs (9) and oscillator model (8) can deliver X_3 in the system (1) to 0 if the phase difference between the oscillators (8) is sufficiently small, that is, $\sin(\theta_1 - \theta_2) \approx \theta_1 - \theta_2$. If the target state X_3^* is not 0, an coordinate transformation $\tilde{X}_3 = X_3 - X_3^*$ enables X_3 to converge to X_3^* .

Let Θ and $\dot{\Theta}$ be $\Theta := \theta_1 - \theta_2$ and $\dot{\Theta} := d\Theta/dt$. The Brockett integrator system (1) with inputs (9) can be rewritten to

$$\begin{aligned} \dot{X}_3 &= -AB\omega_0 \sin \Theta + AB \sin(2\omega_0 t)(K_p\omega_0 X_3 - k \sin \Theta) \\ \dot{\Theta} &= 2(K_p\omega_0 X_3 - k \sin \Theta). \end{aligned}$$

When $\sin \Theta \approx \Theta$, these equations can be linearized to

$$\dot{X}_3 = -AB\omega_0 \Theta + AB \sin(2\omega_0 t)(K_p\omega_0 X_3 - k\Theta) \quad (10)$$

$$\dot{\Theta} = 2(K_p\omega_0 X_3 - k\Theta). \quad (11)$$

The stability condition of the linearized system (10) and (11) is given by the following theorem.

Theorem 1. *The origin $(X_3, \Theta) = (0, 0)$ of (10), (11) is asymptotically stable if $k > ABK_p\omega_0/2$.*

Proof. At first, we introduce Krasovski theorem [13], which gives a sufficient condition for a stability of periodic dynamical system.

Krasovski theorem. *Let $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$ be a differential equation which is periodic in t with period T . Suppose there exists a $\mathbb{C}^1$ function $V : I \times \Omega \rightarrow \mathbb{R}$, periodic in t with period T , such that for some function $a \in \mathbb{R}$ and every $(t, \mathbf{x}) \in I \times \Omega$:*

- 1) $V(t, \mathbf{x}) \geq a(\|\mathbf{x}\|); V(t, 0) = 0;$
- 2) $\dot{V}(t, \mathbf{x}) \leq 0;$
- 3) *except for the origin, M contains no complete positive semi-trajectory where $M = \{(t, \mathbf{x}) \in I \times \Omega : \dot{V} = 0\};$*

are satisfied.

Then, the origin is uniformly asymptotically stable.

Using this theorem, we show the stability of the system (10), (11) .

Let $\mathbf{x} = (X_3, \Theta)$. Here, $V(t, \mathbf{x})$ and $a(\|\mathbf{x}\|)$ are defined as follows,

$$V(t, \mathbf{x}) = 2(K_p\omega_0 X_3 - k\Theta)^2 + ABK_p\omega_0^2 \Theta^2 \quad (12)$$

$$=: \alpha X_3^2 + 2\beta X_3 \Theta + \gamma \Theta^2 \quad (13)$$

$$a(\|\mathbf{x}\|) = \frac{\alpha + \gamma + \sqrt{(\alpha - \gamma)^2 + 4\beta^2}}{2} (X_3^2 + \Theta^2). \quad (14)$$

It is obvious that $V(t, 0) = 0$. From Eqs. (13) and (14), $V(t, \mathbf{x}) - a(\|\mathbf{x}\|)$ yields

$$\begin{aligned}
& V(t, \mathbf{x}) - a(\|\mathbf{x}\|) \\
&= \frac{\alpha - \gamma + \sqrt{(\alpha - \gamma)^2 + 4\beta^2}}{2} X_3^2 + 2\beta X_3 \Theta + \frac{-\alpha + \gamma + \sqrt{(\alpha - \gamma)^2 + 4\beta^2}}{2} \Theta^2 \\
&= \left(\sqrt{\frac{\alpha - \gamma + \sqrt{(\alpha - \gamma)^2 + 4\beta^2}}{2}} X_3 + \sqrt{\frac{-\alpha + \gamma + \sqrt{(\alpha - \gamma)^2 + 4\beta^2}}{2}} \Theta \right)^2 \\
&\geq 0.
\end{aligned} \tag{15}$$

From the inequality, it is verified that the condition 1) in Krasovski Theorem is satisfied.

Next, differentiating $V(t, \mathbf{x})$ with respect to t , $\dot{V}(t, \mathbf{x})$ can be derived as

$$\begin{aligned}
\dot{V} &= 4(K_p \omega_0 X_3 - k\Theta)(K_p \omega_0 \dot{X}_3 - k\dot{\Theta}) + 2ABK_p \omega_0^2 \dot{\Theta} \Theta \\
&= 2\dot{\Theta} \{K_p \omega_0 (-AB\omega_0 \Theta + AB \sin(2\omega_0 t) \dot{\Theta}/2) - k\dot{\Theta}\} + 2ABK_p \omega_0^2 \dot{\Theta} \Theta \\
&= (ABK_p \omega_0 \sin(2\omega_0 t) - 2k) \dot{\Theta}^2
\end{aligned} \tag{16}$$

$$\leq (ABK_p \omega_0 - 2k) \dot{\Theta}^2. \tag{17}$$

From Eq. (17), it can be said that the condition 2) in Krasovski Theorem is satisfied if $k > ABK_p \omega_0/2$.

Finally, from Eqs. (16), (11), M can be described as:

$$M = \{(t, X_3, \Theta) : K_p \omega_0 X_3 - k\Theta = 0\} \tag{18}$$

Suppose a $(t^*, X_3^*, \Theta^*) \in M$. If a trajectory which starts from (t^*, X_3^*, Θ^*) are always contained in M , the equation $K_p \omega_0 \dot{X}_3 - k\dot{\Theta} = 0$ has to be satisfied at least. From Eqs. (10), (11), the time evolution of (X_3, Θ) from (t^*, X_3^*, Θ^*) can be derived as:

$$\dot{X}_3 = \omega_0 \Theta^*, \quad \dot{\Theta} = 0. \tag{19}$$

Therefore, only the origin can stay on M , and then, it is verified that the condition 3) in Krasovski Theorem is satisfied.

The above discussion provides that all conditions of Krasovski Theorem are satisfied if $k > ABK_p \omega_0/2$. Therefore, the origin $\mathbf{x} = (X_3, \Theta) = (0, 0)$ of the system (10), (11) is asymptotically stable if $k > ABK_p \omega_0/2$. $\square$

From the theorem, it can be said that the proposed control method can deliver X_3 to 0 if the phase difference between the oscillators is sufficiently small. To prevent that the phase difference between the oscillators $\theta_1 - \theta_2$ becomes large, it is necessary that the initial phase difference $\theta_1(0) - \theta_2(0)$ is small. The initial phases $\theta_1(0)$ and $\theta_2(0)$ are calculated from $\theta_1(0) = \arcsin(X_1(0)/A)$ and $\theta_2(0) = \arcsin(X_2(0)/B)$ based on relationships $X_1 = \int U_1 dt = A \sin \theta_1$ and $X_2 = \int U_2 dt = B \sin \theta_2$. Consequently, it is necessary that the initial state difference $X_1(0) - X_2(0)$ is sufficiently small for X_3 to be converged to 0 with the proposed control method.

4.1.3 Simulation result

We verified the proposed control method through numerical simulations. Figure 3 shows the simulation results when the initial state was $\mathbf{X} = [0, 0.5, 5]^T$ and the feedback gain was $K_p = 0.1$ in Eq. (8). Other parameters were set as $k = \pi$, $\omega_0 = \pi$ and $A = B = 1$, such that the condition $k > ABK_p \omega_0/2$ was satisfied.

Figure 3(a) shows the trajectories of X_1 , X_2 , and X_3 and Fig. 3(b) shows the phase difference $\theta_1 - \theta_2$. As shown, X_3 successfully converged to 0 and phase difference $\theta_1 - \theta_2$ also converged to 0. Therefore, it was verified that X_3 in Brockett integrator (1), which corresponds to the direction of the Lie bracket, can be controlled by proposed control method (8). Figure 4 shows another simulation result for when the feedback gain was changed to $K_p = 0.3$. In this case, the phase difference converged to 2π , whereas X_3 also successfully converged to 0. Figure 5 shows the simulation results for when $K_p = 0.1$ and $k = 0.01\pi$, which does not satisfy the condition $k > ABK_p \omega_0/2$. In this case, θ_1 and

(a) X_1, X_2, X_3

(b) Phase difference $\theta_1 - \theta_2$

Fig. 3. Control of Brockett integrator with proposed method ($K_p = 0.1$, $k = \pi$).

(a) X_1, X_2, X_3

(b) Phase difference $\theta_1 - \theta_2$

Fig. 4. Control of Brockett integrator with proposed method ($K_p = 0.3$, $k = \pi$).

(a) X_1, X_2, X_3

(b) Phase difference $\theta_1 - \theta_2$

Fig. 5. Control of Brockett integrator with proposed method ($K_p = 0.1$, $k = 0.01\pi$).

θ_2 are not synchronized and X_3 does not converge. These results verify that the proposed method under the condition $k > ABK_p\omega_0/2$ can control the Brockett integrator system. Here, it should be noted that 1) the condition $k > ABK_p\omega_0/2$ is only a sufficient condition, that is, X_3 may converge to 0 even with $k \leq ABK_p\omega_0/2$ depending on a initial state, 2) the choice of k and K_p affects the convergence property as shown in Figs. 3, 4 and 5. The dailed analysis of the convergence property depending on k and K_p is left as a future work.

4.2 Control of two-wheeled vehicle

In this section, we apply the proposed approach to the control of a two-wheeled vehicle, which is one of the nonholonomic systems as well as Brockett integrator.

Fig. 6. Kinematic model of two-wheeled vehicle.

4.2.1 Transformation from Brockett integrator to two-wheeled vehicle

Figure 6 shows a kinematic model of the two-wheeled vehicle. Under nonslip and nonslide assumptions, the kinematic equation of the two-wheeled vehicle model is shown as follows:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \cos \phi/2 & \cos \phi/2 \\ \sin \phi/2 & \sin \phi/2 \\ 1/2r & -1/2r \end{pmatrix} \begin{pmatrix} v_r \\ v_l \end{pmatrix}, \quad (20)$$

where the position (x, y) is the center of gravity of vehicle; v_r and v_l are velocities of the right and left wheels, respectively; ϕ is the attitude of the vehicle; and $2r$ is the vehicle width.

In nonlinear control theory, it is known that a two-wheeled vehicle system is classified into the same class of nonholonomic systems as the Brockett integrator and that these two systems are equivalent under coordinates and input transformations as long as $|\phi| \leq \pi/2$ [10, 11]. Such transformation between Brockett integrator (1) and two-wheeled vehicle model (20) can be derived as follows:

$$X_1 = x, \quad X_2 = \tan \phi, \quad X_3 = 2y - x \tan \phi \quad (21)$$

and

$$v_r = \frac{U_1}{\cos(\tan^{-1} X_2)} + r \cos^2(\tan^{-1} X_2) U_2, \quad (22)$$

$$v_l = \frac{U_1}{\cos(\tan^{-1} X_2)} - r \cos^2(\tan^{-1} X_2) U_2. \quad (23)$$

Under the non-slip and non-slide constraints, the two-wheeled vehicle can not move toward the positive direction of y-axis, which is vertical to the wheels, in a moment. Then, in order to control y of the two-wheeled vehicle (20), we try to control X_3 in Eq. (21) with the proposed control method (8) and (9).

Then, we focus on the equivalence between Brockett integrator and the two-wheeled vehicle system. To control y of the two-wheeled vehicle (20), we attempt to control X_3 in Eq. (21) by using proposed control methods (8) and (9), which were used to control X_3 of Brockett integrator."

4.2.2 Simulation result

We confirmed the effects of the proposed method through numerical simulation. The physical parameter was set as $2r = 0.24$. The control parameters in Eq. (8) were set as $k = \pi$, $\omega_0 = \pi$, $K_p = 0.3$ and $A = B = 1$. Figure 7 shows the simulation results with the initial state $(x, y, \phi)^T = (0, 0, -3)^T$. Figures 7(a) and 7(b) show the trajectories of (x, y, ϕ) and $X_3 = 2y - x \tan \phi$, respectively. Although y continued to oscillate slightly, the trajectory of y successfully converged to near 0 and $X_3 = 2y - x \tan \phi$ converged to 0. From this result, it is verified that the proposed control method (8) can be also applied to the control of y in the two-wheeled vehicle, which corresponds to the vertical direction of the wheels.

5. Feedback control of quadrupedal quasi-passive dynamic walking robot

Finally, we apply the proposed control method to controlling the quadrupedal quasi-passive dynamic walking robot and verify its effectiveness through experiments with our quadrupedal walking robot.

Fig. 7. Control of two-wheeled vehicle with the proposed control method.

5.1 Passive dynamic walking

Passive dynamic walking (PDW) describes a class of mechanical devices that are able to walk down a shallow slope with no actuation [14, 15]. The PDW robot consists of very simple mechanisms. By using its dynamics and gravity, it can achieve very smooth walking, similar to that of a human, with no actuation or control. It is also well known that PDW robots show chaotic behavior [16–18], such as entrainment and bifurcation phenomena. The entrainment phenomenon of PDW means that even if the robot starts walking under different initial conditions, its walking converges to a specific trajectory. Another interesting type of behavior is bifurcation phenomenon. Depending on changes in slope angle or the robot’s parameters, normal (one-period) stable walking bifurcates to two-period walking. Such period doubling, such as bifurcation to four-period from two-period walking or to eight-period from four-period walking, can also occur. These phenomena were verified not only by numerical simulations but also by experiments with real walking robots [18]. Considering such interesting features, much attention has focused on PDW as an approach for producing walking robots [16, 19].

Until now, PDW has primarily been studied with bipedal robots. One reason is that PDW appeared to be a breakthrough in the realization of a smooth bipedal walking robot, which has been considered to be quite difficult. However, there are many walking creatures that have more than four legs. To pursue the principle of legged locomotion, it is interesting and critical to analyze whether a PDW robot with four legs or more is achievable. In fact, several studies are related to quadrupedal PDW [20–22]. Our group developed several multi-legged PDW robots named Jenkka I, II, and III. Jenkka-I has four legs, Jenkka-II has 6-legs and Jenkka-III has 20-legs. The leg number of these robots can be reduced because of their special structure. By using these robots, we verified that not only bipedal PDW but also multi-legged (quadruped, hexapod ... 20-legged) PDW is achievable [22, 23]. With an appropriate initial condition, these robots can excite lateral oscillation on a slope. Depending on its spherical sole shape, each leg can be swung forward in keeping with the excited lateral rocking motion. The lateral oscillation and leg swing motion are constantly repeated, thus these robots achieves locomotion.

5.2 Quadrupedal quasi-passive dynamic walking Robot “Duke”

One research topic on PDW is to apply PDW phenomena to the control of a walking robot on level ground. Contrary to walking down a slope, energy input is essential for continuous walking on level ground. Certain researchers have achieved smooth and high-energy efficiency robot walking by using

the idea of PDW and certain small input torque [19, 24–26]. We call this concept “quasi-passive dynamic walking” [26].

Combining the concept of quasi-PDW and quadrupedal-PDW, we developed a quadrupedal quasi-passive dynamic walking robot, Duke (Fig. 8) [7]. This robot was designed based on Jenkka II and III and was able to walk only on level ground by exerting little energy in the lateral direction using invert pendulums (Fig. 8(a)). Servo motors are located at the roots of the invert pendulums; the motions of these invert pendulums can be controlled separately by the servo motors. The fundamental structural element is a two-legged PDW walker, and the two elements are connected by a “trunk.” The legs of two-legged PDW are linked with passive joints and can swing individually (Fig. 8(b)). The trunk is twistable; that is, the trunk can offer one degree of freedom of relative rotation to the front and rear two-legged elements. Because of this structure, it is feasible to create a phase difference between each two-legged element by using independent inputs to the front and rear servo motors, $\phi_f(t)$, $\phi_h(t)$ (Fig. 8(b)).

In our previous study, we applied Eqs. (24), (25) as periodic inputs to the front and rear servo motors:

$$\phi_f(t) = A \sin\left(\frac{2\pi}{T}t\right), \quad (24)$$

$$\phi_h(t) = A \sin\left(\frac{2\pi}{T}t - \psi\right), \quad (25)$$

where A [rad], T [s] and ψ [rad] are variable parameters referring to amplitude, period, and phase difference, respectively. We verified that Duke can achieve level walking. Additionally, we verified that its walking speed was changed depending on ψ , and that the average walking speed was maximum specifically when $\psi = 3\pi/2$ [rad]. This result suggested a relation between Duke and the nonholonomic constraint system through Eq. (5). As a result of analyses based on the nonlinear control theory, we revealed that Duke had a nonholonomic constraint that corresponds to that of the two-wheeled vehicle and that the motion of the two-wheeled vehicle and Duke could be correlated via

$$\dot{\phi}_f(t) = \frac{1}{r}v_l, \quad \dot{\phi}_h(t) = \frac{1}{r}v_r, \quad (26)$$

where the walking direction of Duke corresponds in the vertical direction to the wheels of the two-wheeled vehicle. We concluded that Duke’s locomotion consisted of Lie bracket motion derived from a leg swinging motion and a sphere rolling motion in the sole, and that the sole shape contributes to the transitions in walking speed, depending on the phase differences in inputs that exerted the lateral oscillation.

These results suggest the potential to control Duke by proposed control method (9), which can control nonholonomic constraint systems, such as Brockett integrator and two-wheeled vehicles. Next, we apply the proposed control method to controlling the walking distance of Duke.

5.3 Experimental Result

We confirmed the effects of the proposed method through walking experiments with the quadrupedal quasi-PDW robot Duke. Let us define the posture angle ϕ and position (x, y) , which is the center of the robot, as shown in Fig. 8(c). To associate with the result in Section 4.2, Duke’s walking direction was coincided with the positive direction of the y -axis. Position (x, y) and posture angle ϕ were measured with a motion capture system. Inputs to servo motors, $\phi_f(t)$ and $\phi_h(t)$ were calculated as follows; 1) (X_1, X_2, X_3) was calculated based on the measured (x, y, ϕ) and Eq. (21), 2) U_1 and U_2 were calculated based on (X_1, X_2, X_3) and Eqs. (8), (9), 3) v_r and v_l were calculated based on U_1, U_2 and Eqs. (22), (23), 4) $\dot{\phi}_f$ and $\dot{\phi}_r$ calculated based on v_r, v_l and Eq. (26), 5) ϕ_f and ϕ_h were calculated by numerical integration of $\dot{\phi}_f$ and $\dot{\phi}_r$. Calculated inputs ϕ_f and ϕ_h were wirelessly transmitted to a microcomputer on Duke. The microcomputer drove servo motors based on ϕ_f and ϕ_h .

Figures 9 and 10 show the experiment results for when the initial state was $(x(0), y(0), \phi(0))^T = (0, -0.5, 0)^T$ and the target state was $y^* = 0$. The parameters were set as $k = 3\pi$, $\omega_0 = 2\pi/1.5$,

Fig. 8. Quasi-passive dynamic walking robot "Duke".

Fig. 9. Snapshot of walking experiment with Quadrupedal quasi-passive dynamic walking Robot "Duke".

$K_p = 2$, $A = 0.1$ and $B = 0.3$. Figure 9 shows a snapshot of experimental results for Duke with proposed control method. Figures 10(a) and 10(b) show walking distance y and trajectory (x, y) , respectively. First, Duke walked forward while repeating lateral oscillation of each two-legged element. Gradually, Duke approached y^* and the phase difference between lateral oscillation and walking velocity decreased. Finally, Duke began to stamp around $y \approx 0.15[\text{m}]$.

From the result, the effectiveness of the proposed control method for the control of Duke was confirmed to some extent. However, some steady-state deviation was observed. This is thought to be due to a leg swinging motion, which was not considered in the two-wheeled vehicle model. To reduce the error, it may be effective to add an integral controller into the proposed controller. This issue is left as a future work.

Fig. 10. Experimental: feedback control of walking distance.

6. Conclusion

In this study, we proposed a feedback control method for nonholonomic constrained systems based on the Kuramoto model and verified the stability of the Brockett integrator system with the proposed control method. Through numerical simulations, we verified that the control method can control the Brockett integrator and two-wheeled vehicle, which are well-known nonholonomic constrained systems. Finally, we applied the proposed method to control the walking distance of our quadrupedal quasi-PDW robot, Duke. As described in Section 5.1, there are only several studies related to quadrupedal PDW compared to the research of bipedal PDW. It is interesting and significant that the result of this paper can contribute to not only the research of nonholonomic constrained systems but also the research of quadrupedal-PDW by focusing on the nonholonomic constraint.

The stability analysis in this paper was limited by the assumption that the phase difference is relatively small. To apply this method to more complicated systems, it should be extended in future work.

Acknowledgments

This work was partially supported by the Grant-in-Aid for Young Scientists (B), No.26870337 from JSPS (the Japan Society for the Promotion of Science) and CREST, JST.

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