The cylinder manifold piecewise linear system: Analysis and implementation

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Abstract: This paper presents a novel autonomous chaotic system defined by a second-order piecewise linear system on the cylinder-type phase space and hysteresis switching. This system can generate super-expanding chaos characterized by a very large positive Lyapunov exponent. Using the piecewise linear one-dimensional return map, the chaos generation is guaranteed theoretically. Presenting a simple test circuit, typical phenomena are confirmed experimentally.

Key Words: switched dynamical systems, chaos, return map

1. Introduction

The manifold piecewise linear system (MPL) is a simple switched dynamical system that can generate chaotic attractors [1–3]. The MPL is defined by a second order piecewise linear system and hysteresis switching. In the history of autonomous chaotic systems, the MPL has been recognized as an important example because of the following facts. First, the dynamics is integrated into a one-dimensional piecewise linear return map and the chaos generation [3–5] can be proved theoretically. Second, the MPL can be implemented by a simple test circuit and chaos generation can be confirmed experimentally. Third, the MPL has been applied to engineering systems: communication systems, signal processor, radar systems, and particle swarm optimizers [6–10].

Since the first autonomous chaotic system, the Lorenz system [11], has been presented, a variety of autonomous chaotic systems have been studied [12–15]. For example, the Chua’s circuit is a simple autonomous circuit that can generate chaos and various bifurcation phenomena. Study of autonomous chaotic systems is important not only as a basic study of nonlinear dynamics but also for engineering applications.

This paper presents a novel kind of MPL: the cylinder manifold piecewise linear system (CMPL).
As with the MPL, the CMPL is defined by a second-order piecewise linear system and hysteresis switching. However, the second-order system is defined on the cylinder-type phase space. It is different from the MPL that is defined in the rectangular coordinate phase space. The dynamics is integrated into a piecewise linear one-dimensional return map on a segment and chaos generation can be guaranteed theoretically. Especially, it is shown that the CMPL can generate super-expanding chaos (SEC) characterized by a very large Lyapunov exponent. Using basic analog/digital circuit elements, a test circuit of the CMPL can be implemented. Using the test circuit, the SEC and chaos can be confirmed experimentally. Since the MPL cannot generate the SEC, results of this paper can contribute to classification of chaotic phenomena and its application to engineering systems. Preliminary results along these lines can be found in our conference papers [16, 17].

2. Manifold piecewise linear system

We recall the MPL [1–3] that is a basic system of the CMPL. The MPL is defined by the following second order piecewise linear system and hysteresis switching:

\[
\ddot{x} - 2\delta \dot{x} + x = \begin{cases} 
p & (+) 
\end{cases} \quad (-), \tag{1}
\]

where \(x\) denotes the dimensionless state variable, \(\tau\) denotes the dimensionless time, and \(\dot{x} \equiv \frac{dx}{d\tau}\). A circuit model of the MPL is discussed in Section 4.

Switching rule of the MPL: Let the right hand side of Eq. (1) be either (+) or (−) at \(\tau = 0\). The right hand side of Eq. (1) is switched from (+) to (−) if a trajectory hits \(L_−\). The right hand side of Eq. (1) is switched from (−) to (+) if a trajectory hits \(L_+\) (see Fig. 1), where

\[
L \equiv L_+ \cup L_-, \quad L_+ \equiv \{ x | x \geq T_h, \dot{x} = 0 \}, \quad L_- \equiv \{ x | x < T_h, \dot{x} = 0 \}, \quad x \equiv (x, \dot{x}).
\]

The MPL is characterized by three parameters: the damping \(\delta\), the equilibrium point \(p\), and the switching threshold \(T_h\). For simplicity, we assume

\[
0 < \delta < 1, \quad 0 < p < 1, \quad T_h = 0.
\]

In this case, Eq. (1) has unstable complex characteristic roots \(\delta \pm j\omega\), where \(\omega \equiv \sqrt{1 - \delta^2}\). As shown in Fig. 1, the trajectories rotate divergently around equilibrium points \(\pm p\). If the trajectory hits negative \(x\)-axis \((L_-)\) then the equilibrium point is switched from \(p\) to \(-p\). If the trajectory hits positive \(x\)-axis \((L_+)\) then the equilibrium point is switched from \(-p\) to \(p\). Note that the switching occurs only on \(x\)-axis \((L)\). Repeating the rotation and switching, the MPL can exhibit chaotic trajectories as shown in Figs. 2(a) and (b). The trajectories can be calculated by the piecewise exact solution.
Fig. 2. Trajectories of the manifold piecewise linear system for \( p = 1 \). (a) \( \beta = \sqrt{2} \) (b) \( \beta = 1.8 \) (c) \( \beta = 2.6 \).

Fig. 3. Return maps of MPL for \( p = 1 \). (a) \( \beta = \sqrt{2} \), (b) \( \beta = 1.8 \), (c) \( \beta = 2.6 \).

\[
x = p + (x(0) - p)e^{\frac{\delta \pi}{\omega}} \cos \tau + (\dot{x}(0) - \delta(x(0) - p))e^{\frac{\delta \pi}{\omega}} \sin \tau,
\]

where \((x(0), \dot{x}(0))\) is an initial condition at \( \tau = 0 \).

In order to define the return map, we consider a trajectory started from a point \( x_0 \in L \) at \( \tau = 0 \) where a point on \( L \) is represented by its \( x \) coordinate. The trajectory intersects \( L \) at \( \tau = \pi/\omega \) and let \( x_1 \) be the intersection. Since \( x_0 \) determines \( x_1 \), we can define the 1D return map \( F \) from \( L \) to itself. The map is piecewise linear and is described exactly [1–3]:

\[
x_1 = F(x_0) \equiv \begin{cases} 
-\beta(x_0 - p) + p & \text{for } x_0 \geq 0 \\
-\beta(x_0 + p) - p & \text{for } x_0 < 0,
\end{cases}
\]

where \( \beta \equiv e^{\frac{\delta \pi}{\omega}} \) (we will use either \( \beta \) or \( \delta \) for convenience of explanation). Now the dynamics is integrated into the iteration \( x_{n+1} = F(x_n) \). The chaos generation is guaranteed if \( 1 < \beta < 2 \). In this case, there exists an invariant interval \( I_1 \) on which the map is expanding:

\[
F(I_1) \subseteq I_1, \ |DF(x)| > 1, \text{ for } x \in I_1 \equiv (-F(0), F(0)),
\]

where \( DF(x) \) is the slope of \( F \) at \( x \). The map has a positive Lyapunov exponent \( \ln \beta \) [3, 4]. Figure 3 shows return maps corresponding to Fig. 2. Note that trajectories diverge for \( \beta > 2 \).

3. Cylinder manifold piecewise linear system

For convenience, we define the MPL with the infinite number of equilibria (MPL\(_\infty\)) that is equivalent to the CMPL:

\[
\ddot{x} - 2\delta \dot{x} + x = \begin{cases} 
+ \frac{p + 2nT}{(n+)} \\
- \frac{p + 2nT}{(n-)}
\end{cases}
\]
Switching rule of the MPL\(_\infty\): Let the right hand side of Eq. (5) be either \((n+)\) or \((n-)\) for some \(n\) at \(\tau = 0\). The right hand side is switched to \((n+)\) if the trajectory hits \(L_{n^+}\). The right hand side is switched to \((n-)\) if the trajectory hits \(L_{n^-}\), where

\[
L_n \equiv L_{n^+} \cup L_{n^-}, \quad L_{n^+} \equiv \{x \mid 2nT < x \leq (2n+1)T, \dot{x} = 0\} \\
L_{n^-} \equiv \{x \mid (2n-1)T < x \leq 2nT, \dot{x} = 0\},
\]

\(n\) denotes integers, \(0 < \delta < 1\), and \(0 < p < 1\). The trajectory rotates around either of the equilibrium points \(p + 2nT\) and \(-p + 2nT\) as shown in Fig. 4(a). If the trajectory hits \(L_{n^+}\), the equilibrium point is switched to \(p + 2nT\). If the trajectory hits \(L_{n^-}\), the equilibrium point is switched to \(p - 2nT\).

Identifying \(L_n\) with \(L_0\) for all \(n\), the CMPL is defined. The identification is represented by

\[
L_0 = G(L_n), \quad G(X) = ((X + T) \mod 2T) - T. \tag{6}
\]

The mapping \(G\) constructs the cylinder-type phase space. The CMPL is defined by the following.

\[
\ddot{x} - 2\delta \dot{x} + x = \begin{cases} 
p & (+) \\
-p & (-).
\end{cases} \tag{7}
\]

Switching rule of the CMPL: Let the right hand side of Eq. (7) be either (+) or (−) at \(\tau = 0\). The right hand side is switched to (+) (respectively, (-)) such that

\[
x(\tau_+), \dot{x}(\tau_+) = (G(x(\tau)), 0), \quad \tau_+ \text{ is a time just after } \tau,
\]

if the trajectory hits \(L_{n^+}\) (respectively, \(L_{n^-}\)) at time \(\tau\). The switching is illustrated in Fig. 4(b).

The CMPL is characterized by three parameters: damping \(\delta\), the equilibrium point \(p\), and the cylinder circumference \(2T\). For simplicity, we assume the following:

\[
T = 2, \quad 0 < p < T, \quad 0 < \delta < 1, \quad (1 < \beta \equiv e^{\frac{2\pi}{T}} < \infty).
\]

The CMPL generates chaotic trajectories as shown in Fig. 5.

Let us introduce the return map of the CMPL. Since a trajectory started from \(L_0\) must return to \(L_0\), the dynamics of the CMPL can be integrated into the 1D return map from \(L_0\) to itself. The map is exactly piecewise linear and is described by

\[
x_1 = f(x_0) \equiv G(F(x_0)), \quad F(x) = \begin{cases} 
-\beta(x - p) + p & \text{for } x \in L_{1^+} \\
-\beta(x + p) - p & \text{for } x \in L_{1^-}.
\end{cases} \tag{8}
\]

Fig. 4. (a) Switching of MPL\(_\infty\). The green and blue segments denote \(L_{n^+}\) and \(L_{n^-}\), respectively. (b) Switching of the CMPL.
Fig. 5. Trajectories of CMPL for $p = 1$ and $T = 2$. (a) $\beta = \sqrt{2}$, (b) $\beta = 1.8$, (c) SEC for $\beta = 2.3$, (d) SEC for $\beta = 2.4$, (e) SEC for $\beta = 2.5$, (f) SEC for $\beta = 2.6$. 
Fig. 6. Return maps of CMPL for $p = 1$ and $T = 2$. (a) $\beta = \sqrt{2}$, (b) $\beta = 1.8$, (c) $\beta = 2.3$, (d) $\beta = 2.4$, (e) $\beta = 2.5$, (f) $\beta = 2.6$. 
Note that the domain of the map $L_0$ is an invariant interval: $f(L_0) \subseteq L_0$. Hence the trajectory does not diverge as far as $\beta$ is finite. This is the major difference from the MPL domain of whose return map is $x$-axis. Figure 6 shows examples of the return map corresponding to Fig. 5. Since the slope of map is constant ($\beta$), the Lyapunov exponent $\lambda$ of the map is $\ln \beta > 0$ [4]. We can see that the CMPL can generate chaos with a large Lyapunov exponent $\lambda > \ln 2$. We refer to the chaos with $\lambda > \ln 2$ as the SEC. Note that, in the MPL for $2 < \beta$, trajectories diverge and the SEC cannot be generated.

4. Laboratory experiments

First, we introduce a test circuit of the MPL in Fig. 7. The test circuit of the CMPL is based on this circuit. The second order circuit is fabricated by op amps (TL072) and the equilibrium point is determined by the switch $S$ (TC4066). The supply voltages of the discrete elements are $\pm 8$[V]. The dynamics is described by

$$\frac{dv_1}{dt} = -v_2, \quad \frac{dv_2}{dt} = \frac{R_0}{R_2}v_2 + \frac{R_0}{R_1}v_1 + \left\{ \begin{array}{ll}
-\frac{R_0E}{R_s} & \text{for } S = (+) \\
+\frac{R_0E}{R_s} & \text{for } S = (-).
\end{array} \right.$$ (9)

The $S$ is switched from $(-)$ to $(+)$ (respectively, $(+)$ to $(-)$) if $v_1 < 0$ and $-v_2 > 0$ (respectively, $v_1 \geq 0$ and $-v_2 < 0$). This rule is equivalent to the rule of MPL with $V_{TH} = 0$. Figure 7(b) shows the control circuit of the switch $S$. If $v_1 < 0$ and $-v_2 > 0$ then the AND (HD4081) of two comparators (TL072) triggers the set terminal of the flip-flop (HD4013). The output connects the switch $S$ to $(+)$ terminal. If $v_1 > 0$ and $-v_2 < 0$ then the AND of two comparators triggers the reset terminal of the flip-flop that connects the switch $S$ to $(-)$ terminal. Repeating the switching, the circuit exhibits chaotic attractor. Figure 8 shows laboratory measurements of chaotic attractors that correspond to Figs. 2(a) and (b).

Using the following dimensionless variables and parameters

$$\tau = \sqrt{\frac{R_0}{R_1} RC}, \quad x = \frac{v_1}{E}, \quad \dot{x} = \frac{dx}{d\tau}, \quad y = \sqrt{\frac{R_1}{R_0} \frac{v_1}{E}}, \quad 2\delta = \sqrt{\frac{R_1 R_0}{E/R_s}}, \quad p = \frac{R_1}{R_s}, \quad T = \frac{V_B}{E}.$$

Equation (9) is transformed into the following equation that is equivalent to Eq. (1).

$$\dot{x} = -y, \quad \dot{y} - 2\delta y - x = \left\{ \begin{array}{ll}
-p & \text{for } S = (+) \\
+p & \text{for } S = (-).
\end{array} \right.$$ (10)

Figure 9 shows a test circuit of the CMPL. The second-order circuit is controlled by four kinds of switches $S$, $S_1$, $S_2$, and $S_3$. For simplicity, we have fabricated this circuit for the case where the switching occur in the range $-2V_B < v_1 < 2V_B$ where $V_B$ is a border voltage value corresponding to the semicircular of the cylinder $T = V_B/E$. It corresponds to the case $1 < \beta < 3$ where the switching occurs in the range $x \in L_{-1+} \cup L_{0-} \cup L_{0+} \cup L_{1-}$. It is possible to extend the range, however, the circuit becomes more complicated. Figure 9(d) shows the control circuit of the switch $S$ that determines the equilibrium point: the right hand side of Eq. (9). The set terminal of the flip flop is triggered and $S$ is connected to terminal $(+)$ if

$$(-V_B < v_1 < 0 \text{ and } -v_2 \geq 0) \text{ or } (V_B < v_1 \text{ and } -v_2 \leq 0).$$

This condition is judged by comparators, AND, and OR circuits. The reset terminal of the flip flop is triggered and $S$ is connected to terminal $(-)$ if

$$(0 < v_1 < V_B \text{ and } -v_2 \leq 0) \text{ or } (v_1 < -V_B \text{ and } -v_2 \geq 0).$$

Figures 9(a) to (c) show the control circuit of switches $S_1$ to $S_3$ that reset of the capacitor voltage $v_1$ into $-V_B < v_1 < V_B$. The $S_2$ is closed and $S_1$ resets the capacitor voltage $v_1$ to $v_1 - 2V_B$ if $V_B < v_1$ and $-v_2 \leq 0$. 

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Fig. 7. Test circuit of the MPL.

Fig. 8. Laboratory measurements of the MPL for $E \simeq 0.4\,\text{[V]}$, $R \simeq 1\,\text{[k}\Omega\text{]}$, $R_0 \simeq 1\,\text{[k}\Omega\text{]}$, $R_s \simeq 5.1\,\text{[k}\Omega\text{]}$, $C \simeq 0.033\,\text{[\mu}\text{F}\text{]}$. $V_{Th} \simeq 0\,\text{[V]}$ (a) Chaos for $R_1 \simeq 4.9\,\text{[k}\Omega\text{]}$, $R_2 \simeq 10\,\text{[k}\Omega\text{]}$ ($\beta \simeq \sqrt{2}$) (b) Chaos for $R_1 \simeq 3.5\,\text{[k}\Omega\text{]}$, $R_2 \simeq 5.1\,\text{[k}\Omega\text{]}$ ($\beta \simeq 1.8$).
Fig. 9. Test circuit of the CMPL. (a) to (d): Control circuits of switches. M.M. denotes the monostable multivibrator that shapes a pulse signal.
Fig. 10. Laboratory measurements of the CMPL. $R \approx 1[k\Omega]$, $r \approx 1[k\Omega]$, $r_1 \approx 1[k\Omega]$, $r_2 \approx 1[k\Omega]$, $R_0 \approx 1[k\Omega]$, $R_1 \approx 1[k\Omega]$, $C \approx 0.033[\mu F]$, $C_h \approx 1[n F]$, $E \approx 0.4[V]$, $V_B \approx 0.8[V]$ ($T \equiv V_B/E \approx 2$). (a) Chaos. $R_1 \approx 2[k\Omega]$, $R_2 \approx 7[k\Omega]$ ($\beta \approx \sqrt{2}$) (b) Chaos. $R_1 \approx 2[k\Omega]$, $R_2 \approx 3.8[k\Omega]$ ($\beta \approx 1.8$) (c) SEC. $R_1 \approx 2.3[k\Omega]$, $R_2 \approx 3[k\Omega]$ ($\beta \approx 2.3$) (d) SEC. $R_1 \approx 2.3[k\Omega]$, $R_2 \approx 2.8[k\Omega]$ ($\beta \approx 2.4$) (e) SEC. $R_1 \approx 2.3[k\Omega]$, $R_2 \approx 2.7[k\Omega]$ ($\beta \approx 2.5$) (f) SEC. $R_1 \approx 2.3[k\Omega]$, $R_2 \approx 2.6[k\Omega]$ ($\beta \approx 2.6$).
The $S_3$ is closed and $S_1$ resets the capacitor voltage $v_1$ to $v_1 + 2V_B$ if

$$v_1 < -V_B \text{ and } -v_2 \geq 0.$$ 

Figure 10 shows the laboratory measurements of chaotic attractors that correspond to numerical chaotic attractors in Fig. 5. We have confirmed the SEC and chaotic attractors.

### 5. Conclusions

The CMPL is presented and the chaotic dynamics is analyzed in this paper. The dynamics is integrated into a piecewise linear one-dimensional return map and generation of super-expanding chaos is guaranteed theoretically. Presenting the test circuit, typical phenomena are confirmed experimentally.

Future problems include classification of chaotic attractors, laboratory measurements of various phenomena, and engineering applications.

### References


