Ergodic transformations on $\mathbb{R}$ preserving Cauchy laws

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Abstract: We consider a family of ergodic transformations on the real line $\mathbb{R}$ preserving Cauchy laws. A dualistic nature between the ergodic transformation and the associated transformation of the scale parameter of a Cauchy law is proven to be hold, which provides a systematic view of explicit mixing property with the ergodic transformation having the Cauchy law as the limiting distribution.

Key Words: ergodic transformations, Cauchy law, mixing property, stable law, addition theorem

1. Introduction

Ergodic transformations on the real line $\mathbb{R}$ having the power law are the topics of great interest. They explain the origin of the power law behavior which is widely observed in nature and social systems. Recalling that fading probability characteristic in communications channels and price movements of financial markets are generally heavy-tailed, they can serve good random-number generators for Monte Carlo simulations of such practical applications with power law. However, few studies have been made except the studies [1, 2, 8]. Here we study ergodic transformations on the real line $\mathbb{R}$ having Cauchy law and their mixing behavior. This paper is constructed as follows. In Section 2, we introduce ergodic transformations on the real line $\mathbb{R}$ and prove the basic properties. In Section 3, we introduce our theorems on preservation of Cauchy laws and prove the theorems. Finally, we conclude this paper.

2. Ergodic transformations on the real line $\mathbb{R}$

The following simple ergodic transformation $T : \mathbb{R} \to \mathbb{R}$ defined by

$$T0 := 0, \quad Tx := \frac{1}{2}(x - \frac{1}{x}) \quad \text{for} \quad x \neq 0$$

appears in many literatures [1, 7–10]. Consider more generic ergodic transformation $B_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ which is defined by

$$B_{\alpha,\beta}(0) := 0, \quad B_{\alpha,\beta}(x) := \alpha x - \frac{\beta}{x} \quad \text{for} \quad x \neq 0.$$ 

In 1857, G. Boole [4] obtained the Boole transformation $B_{1,1} : \mathbb{R} \to \mathbb{R}$ defined by

$$B_{1,1}(0) := 0, \quad B_{1,1}(x) := x - \frac{1}{x} \quad \text{for} \quad x \neq 0.$$ 

which has the invariance property for an integrable function \( f(x) \) and the Boole transformation \( T = B_{1,1}(x) \) as follows:

\[
\int_{\mathbb{R}} f(Tx) \, dx = \int_{\mathbb{R}} f(x) \, dx.
\]

This means the Boole transformation has the infinite invariant measure \( \mu(dx) \) given by the Lebesgue measure. In 1973, Adler and Weiss proved the ergodicity of the Boole transformation [3] and later on, the ergodicity of more generalized Boole transformation such as \( T = \frac{1}{2}(x - \frac{1}{x}) \) was shown in [1, 8]. In 1998, the present author [1] showed ergodicity and mixing property of such transformations on \( \mathbb{R} \) given by the addition theorem of cot \( \theta \) and tan \( \theta \) according to the general method of constructing exactly solvable chaos [11]. Here, we introduce a family of an infinite number of ergodic transformations \( F_K : \mathbb{R} \to \mathbb{R} \) defined by the addition theorem of cot \( \theta \):

\[
F_K(\cot \theta) := \cot K\theta,
\]

(1)

where \( K(\geq 2) \) is a positive integer. Note that the particular transformations are:

\[
F_2(x) = \frac{1}{2}(x - \frac{1}{x}), \quad F_3(x) = \frac{x^3 - 3x}{3x^2 - 1}, \quad F_4(x) = \frac{x^4 - 6x^2 + 1}{4x^3 - 4x}, \quad \ldots
\]

(2)

In the similar way, we introduce ergodic transformations \( I_K : \mathbb{R} \to \mathbb{R} \) defined by the addition theorem of tan \( \theta \):

\[
I_K(\tan \theta) := \tan K\theta,
\]

(3)

where \( K(\geq 2) \) is a positive integer. Note that those counterpart transformations are:

\[
I_2(x) = \frac{2x}{1 - x^2}, \quad I_3(x) = \frac{3x - x^2}{1 - 3x^2}, \quad I_4(x) = \frac{4x - 4x^4}{1 - 6x^2 + x^4}, \quad \ldots
\]

(4)

All the transformations \( F_K, I_K \) at \( K \geq 2 \) have mixing property (thus ergodic) [1] with the positive Lyapunov exponents \( \lambda(= \log K) \), which equals to its Kolmogorov-Sinai entropy \( h_\mu \) due to the Pesin identity with respect to the ergodic invariant measure \( \mu \) given by the Cauchy density on \( \mathbb{R} \):

\[
\mu(dx) = \rho(x) \, dx = \frac{dx}{\pi(1 + x^2)}.
\]

(5)

The key to proving this fact is simple: all the transformations \( F_K, I_K \) at \( K \geq 2 \) have the topological conjugacy relation with the piecewise linear map with the absolute value of the slope being \( K \) by a diffeomorphism. Thus, by the pointwise ergodic theorem of Birkhoff [5], we have the following theorem:

**Theorem (Basic Ergodicity)** Consider a map \( T : \mathbb{R} \to \mathbb{R} \), which is given by \( F_K \) or \( I_K \) at \( K \geq 2 \). For an integrable function \( f \), the following relation holds for almost everywhere \( x \in \mathbb{R} \):

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \int_{\mathbb{R}} f(y) \frac{dy}{\pi(1 + y^2)}.
\]

Here, \( T^n x \) is the \( n \) times iteration of the mapping \( T \) of \( x \).

Furthermore, when we set \( f(x) = e^{ikx} \) with \( k \in \mathbb{R} \), we have the following relation regarding the Chaos-Fourier Transform computing the Fourier Transform by Chaotic Monte Carlo method [6] based on the pointwise ergodic theorem with chaotic maps with absolutely continuous invariant measure \( \mu(dx) = \rho(x) \, dx \).

**Theorem (Chaos Fourier Transform)** Consider a map \( T : \mathbb{R} \to \mathbb{R} \), which is given by \( F_K \) or \( I_K \) at \( K \geq 2 \). The following relation holds for almost everywhere \( x \in \mathbb{R} \):
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{ikT^nx} = e^{-|k|}.
\]

**Proof**

With an integrable function \( f(x) = e^{ikx} \) with \( k \in \mathbb{R} \), the pointwise ergodic theorem states:

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{ikT^nx} = \int_{\mathbb{R}} e^{iky} \mu(dy) = \int_{\mathbb{R}} e^{iky} \frac{dy}{\pi(1+y^2)} = e^{-|k|}
\]

for almost everywhere \( x \). The last equality holds because of the well-known mathematical fact that the characteristic function \( \phi(k) \) of the Cauchy density function \( \rho(x) = \frac{1}{\pi(1+x^2)} \) is the function of the exponential decay:

\[
\int_{\mathbb{R}} e^{ikx} \frac{dx}{\pi(1+x^2)} = e^{-|k|}.
\]

More generally, the time average of \( e^{ikX^nx} \) on \( \mathbb{R} \) naturally computes the associated characteristic function \( \phi(k) \) of the Fourier spectrum \( k \) for an ergodic invariant measure \( \mu(dx) \) on \( \mathbb{R} \) by this chaos Fourier transform method just utilizing the pointwise ergodic theorem of Birkhoff. This is the fundamental property of ergodic transformations on \( \mathbb{R} \). Recently, Steuding [7] found a remarkable relation between the Riemann hypothesis on the Riemann zeta function \( \zeta(s) \) and the ergodic transformation \( Tx = F_2(x) = \frac{1}{2}(x - \frac{1}{2}) \) by using this type of ergodic theorem:

**Theorem (Steuding [7], 2012)** For almost all \( x \in \mathbb{R} \) and the ergodic transformation \( Tx = \frac{1}{2}(x - \frac{1}{2}) \) on \( \mathbb{R} \),

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \log \left| \zeta\left(\frac{1}{2} + \frac{1}{2}iT^nx\right) \right| = \sum_{\text{Re} \phi > \frac{1}{2}} \log \left| \frac{\rho}{1-\rho} \right|;
\]

in particular, the Riemann hypothesis is true if, and only if, one and thus either side vanishes, the left-hand side for almost all real \( x \).

Here, \( \rho \) is the non-trivial zeros of \( \zeta(s) \). The key of proving the above theorem is based on the fact that \( Tx = F_2(x) = \frac{1}{2}(x - \frac{1}{2}) \) is ergodic with respect to the invariant measure \( \mu(dx) = \frac{dx}{\pi(1+x^2)} \).

Thus, we obtain the similar theorem related to an infinite number of ergodic transformations on \( \mathbb{R} \) by using the fact that \( Tx = F_K(x) \) or \( Tx = I_K(x) \) on \( \mathbb{R} \) with \( K \geq 2 \) is ergodic with respect to the invariant measure \( \mu(dx) = \frac{dx}{\pi(1+x^2)} \) as follows.

**Theorem (Riemann Hypothesis and Ergodic Transformations on \( \mathbb{R} \))** For almost all \( x \in \mathbb{R} \) and the ergodic transformations \( Tx = F_K(x) \) or \( Tx = I_K(x) \) on \( \mathbb{R} \) with \( K \geq 2 \),

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \log \left| \zeta\left(\frac{1}{2} + \frac{1}{2}iT^nx\right) \right| = \sum_{\text{Re} \phi > \frac{1}{2}} \log \left| \frac{\rho}{1-\rho} \right|;
\]

in particular, the Riemann hypothesis is true if, and only if, one and thus either side vanishes, the left-hand side for almost all real \( x \).

That is an interesting relation between the Riemann hypothesis and ergodic transformations on \( \mathbb{R} \).

3. Preservation of Cauchy laws

In this section, we further investigate about how the Cauchy distribution \( \rho_\lambda(x)dx \) with the scale parameter \( \lambda > 0 \) is affected by a change of variable by the ergodic transformations on \( \mathbb{R} \). Let us consider the density function of the Cauchy law with the scale parameter \( \lambda > 0 \):

\[
\]
Here, we have the following theorem.

**Theorem 1 (The Cauchy Invariance for $Tx = F_2(x)$)** Assume that a variable $X \in \mathbb{R}$ obeys the Cauchy law with the density $\rho_\lambda(x)dx$ for $\lambda > 0$. Then, the probability density function of the variable $Y = TX = F_2(X)$ obeys the Cauchy law with the scale parameter $\lambda' = \frac{1}{2}(\lambda + \frac{1}{\lambda})$; namely, the following relation holds:

$$
\rho_\lambda \mapsto \rho_{\lambda'} = \rho_{\frac{1}{2}(\lambda + \frac{1}{\lambda})} \quad \text{when} \quad x \mapsto Tx = F_2(x) = \frac{1}{2}(x - \frac{1}{x}).
$$

Thus, after $n$ times iterations of $T$, we have the semi-group property of the scale parameter of the Cauchy Law:

$$
\rho_\lambda \mapsto \rho_{\lambda_n} = \rho_{G^n(\lambda)} \quad \text{when} \quad x \mapsto T^n x = F_2^n(x),
$$

where $G$ is defined by

$$
G(\lambda) = \frac{1}{2}(\lambda + \frac{1}{\lambda}).
$$

Note that

$$
\lambda_\infty = \lim_{n \to \infty} \lambda_n = \lim_{n \to \infty} G^n(\lambda) = 1
$$

for any $\lambda > 0$ because $\lambda_\infty$ is the unique fixed point of the equation $\lambda = G(\lambda)$ for $\lambda > 0$ and the map $G$ corresponds to one iteration process of the Newton method solving the equation $\lambda^2 - 1 = 0$ with the quadratic convergence speed [10]. This is consistent with the fact the ergodic transformation $Tx = F_2(x)$ has the standard Cauchy law with the absolutely continuous density $\rho_1(x) = \frac{1}{\pi(1 + x^2)}$ as the invariant measure and due to the ergodic and mixing property, the limiting distribution converges to the invariant measure $\mu(dx) = \rho_1(x)dx$ for almost all distributions of the initial value $x$.

**Proof.** (Proof of Theorem 1)

For an arbitrary $y \neq 0$, there are always two solutions $x_1, x_2 \in \mathbb{R}$ satisfying the equation

$$
y = F_2(x) = \frac{1}{2}(x - \frac{1}{x}).
$$

We denote them as $x_1$ and $x_2$ respectively. Then, we have the relation

$$
x_1 + x_2 = 2y, \quad x_1x_2 = -1.
$$

Let us denote the density function of a variable $y = Tx = F_2(x)$ by $p_Y(y)$. The probability $p_Y(y)dy$ equals the sum of the probabilities $\rho_\lambda(x_1)|dx|_{x=x_1}$ and $\rho_\lambda(x_2)|dx|_{x=x_2}$. Thus, we have the following probability preservation relation:

$$
p_Y(y)dy = \rho_\lambda(x_1)|dx|_{x=x_1} + \rho_\lambda(x_2)|dx|_{x=x_2}.
$$

Remark that

$$
\left| \frac{dy}{dx} \right|_{x=x_j} = \left| \frac{dF_2(x)}{dx} \right|_{x=x_j} = \frac{1 + x_j^2}{x_j^2} = \frac{2(1 + y^2)}{1 + x_j^2}, \quad \text{for} \quad j = 1 \quad \text{or} \quad 2,
$$

we obtain $p_Y(y)$ by the relation

$$
p_Y(y) = \frac{1}{2(1 + y^2)} \sum_{j=1}^{2} \rho_\lambda(x_j)(1 + x_j^2) = \frac{1}{2(1 + y^2)} \lambda((1 + x_1^2)(\lambda^2 + x_2^2) + (1 + x_2^2)(\lambda^2 + x_1^2)) = \frac{1}{2(1 + y^2)} \frac{\lambda^2 + 1}{\pi((1 + \lambda^2)^2 + 4y^2\lambda^2)} = \rho_{\frac{1}{2}(\lambda + \frac{1}{\lambda})}(y) = \rho_{G(\lambda)}(y).
$$
Thus, we can say that the generalized Boole transformation \(Tx = F_2(x) = \frac{1}{2}(x + \frac{1}{x})\) preserves the Cauchy law with the scale parameter \(\lambda \mapsto G(\lambda) = \frac{1}{2}(\lambda + \frac{1}{\lambda})\).

Here, we define \(G_K\) as an addition theorem of \(\cot \theta\) satisfying \(G_K(\cot \theta) = \cot K\theta\).

**Theorem 2 (The Cauchy Invariance for \(Tx = F_K(x)\) at \(K \geq 2\))** Assume that a variable \(X \in \mathbb{R}\) obeys the Cauchy law with the density \(\rho_\lambda(x)dx\) for \(\lambda > 0\). Then, the probability density function of the variable \(Y = TX = F_K(X)\) at \(K \geq 2\) obeys the Cauchy law with the scale parameter \(\lambda' = G_K(\lambda)\); namely, the following relation holds:

\[
\rho_\lambda \mapsto \rho_{\lambda'} = \rho_{G_K(\lambda)} \quad \text{when} \quad x \mapsto Tx = F_K(x).
\]

When \(K = 2\), \(\lambda' = G_2(\lambda) = \frac{1}{2}(\lambda + \frac{1}{\lambda})\). Thus, Theorem 2 for \(K = 2\) corresponds to Theorem 1. Thus, Theorem 2 can be regarded as the generalization of Theorem 1.

**Proof. (Proof of Theorem 2)**

Consider a change of variable as \(Y = F_K(X)\). For a given \(Y = y \in \mathbb{R}\), there are \(K\) solutions satisfying the equation \(y = F_K(x)\). We denote the solutions by \(x_1, x_2, \ldots, x_K\). By noting that \(F_K\) can be seen as \(K\)-th multiplication formula of \(\cot \theta\), we can write the solutions explicitly as \(x_j = -\cot(\phi + j \frac{\pi}{K}), j = 1, \ldots, K\) for \(y = -\cot K\phi\) \((0 \leq K\phi \leq \pi)\). Denote the density function of a variable \(y = Tx = F_K(x)\) by \(p_Y(y)\). Then we have the following probability preservation relation:

\[
p_Y(y)dy = \rho_{\lambda}(x_1)|dx|_{x=x_1} + \rho_{\lambda}(x_2)|dx|_{x=x_2} + \cdots + \rho_{\lambda}(x_K)|dx|_{x=x_K}.
\]

It is easy to check that the relation holds:

\[
\left| \frac{dy}{dx} \right|_{x=x_j} = \left| \frac{dF_K(x)}{dx} \right|_{x=x_j} = \frac{d}{d\phi}(-\cot K\phi) \frac{d\phi}{dx} = \frac{K}{\sin^2(K\phi)}\sin^2(\phi + j \frac{\pi}{K}) = \frac{K(1+y^2)}{1+x_j^2}.
\]

Thus, \(p_Y(y)\) satisfies the relation

\[
p_Y(y) = \frac{1}{K(1+y^2)} \sum_{j=1}^{K} \rho_{\lambda}(x_j)(1+x_j^2) = \frac{1}{K\pi(1+y^2)} \sum_{j=1}^{K} \frac{\lambda}{1+(\lambda^2-1)\sin^2(\phi + j \frac{\pi}{K})}.
\]

By noting that the following identity

\[
\sum_{j=1}^{K} \frac{\lambda}{1+(\lambda^2-1)\sin^2(\phi + j \frac{\pi}{K})} = \frac{K G_K(\lambda)(1 + \cot^2(K\phi))}{G_K(\lambda) + \cot^2(K\phi)}
\]

always holds, we obtain \(p_Y(y)\) as

\[
p_Y(y) = \frac{1}{K\pi(1+y^2)} \sum_{j=1}^{K} \frac{\lambda}{1+(\lambda^2-1)\sin^2(\phi + j \frac{\pi}{K})} = \frac{1}{K\pi(1+y^2)} \frac{K G_K(\lambda)(1+y^2)}{(G_K(\lambda) + y^2)} = \rho_{\lambda'}(y),
\]

where \(\lambda' = G_K(\lambda)\).

After \(n\)-th iteration of the ergodic transformation \(Tx = F_K(x)\), the density function obeys the semi-group property as \(\rho_{\lambda_n} = \rho_{G_K^n(\lambda)}\) approaching the standard Cauchy density \(\rho_1(x) = \frac{1}{\pi(1+x^2)}\) of the unique invariant invariant measure \(\mu(dx)\) on the real line \(\mathbb{R}\) as the limiting distribution for \(n \to \infty\).

Letac [9] showed that generalized Boole transformations including \(Tx = F_2(x)\) on the real line \(\mathbb{R}\) preserve Cauchy laws. While Theorem 1 is considered as a special case of the theorem of Letac, Theorem 2 generalizing Theorem 1 covers wider ergodic transformations \(F_K\) at \(K \geq 2\) (not necessary
the generalized Boole transformations) preserving Cauchy laws as compared to the existing articles on preservation of Cauchy laws such as [8, 9]. Thus, Theorem 2 states a new preservation law of Cauchy laws. Furthermore, we can proceed further as we treat ergodic transformations $I_K$ at $K \geq 2$ satisfying the functional relation $\tan K\theta = I_K(\tan \theta)$. Here we consider rational mappings $J_K$ at $K \geq 2$ which satisfy the relation $\tanh K\theta = J_K(\tanh \theta)$. We have the following theorem.

**Theorem 3 (The Cauchy Invariance for $Tx = I_K(x)$ at $K \geq 2$)** Assume that a variable $X \in \mathbb{R}$ obeys the Cauchy law with the density $\rho_\lambda(x) dx$ for $\lambda > 0$. Then, the probability density function of the variable $Y = TX = I_K(X)$ at $K \geq 2$ obeys the Cauchy law with the scale parameter $\lambda'' = J_K(\lambda)$; namely, the following relation holds:

$$\rho_\lambda \mapsto \rho_\lambda'' = \rho_{J_K(\lambda)} \quad \text{when} \quad x \mapsto Tx = I_K(x).$$

**Proof. (Proof of Theorem 3)**
We note that distribution of $\frac{1}{X}$ where $X$ obeys the Cauchy law with the scale parameter $\lambda$, obeys the Cauchy law with the scale parameter $\frac{1}{\lambda}$. Thus, by Theorem 2,

$$\lambda'' = \frac{1}{G_K(\frac{1}{\lambda})} = J_K(\lambda).$$

4. Conclusion
An infinite number of ergodic transformations $F_K$ and $I_K$ given by the addition theorems of $\cot \theta$ and $\tan \theta$ at $K \geq 2$ are proven to preserve Cauchy laws. The case that $F_2(x) = \frac{1}{2}(x - \frac{1}{x})$ corresponds to the well-known case having the standard Cauchy law as the ergodic invariant measure. The scale parameter $\lambda$ dynamics is also obtained by mappings $G_K$ and $J_K$ given by the addition theorems of $\coth \theta$ and $\tanh \theta$ respectively. Thus, explicit mixing behavior of the probability distribution of $F_K$ and $I_K$ towards the limiting standard Cauchy law with the density $\rho_1(x)$ is systematically obtained by this scale parameter dynamics $G_K$ and $J_K$ respectively.

References


