Paper

Hirota bilinear approach to GUE, NLS, and Painlevé IV

Saburo Kakei^{1 a)}

¹ Department of Mathematics, Rikkyo University,
3-34-1 Nishi-ikebukuro, Toshima-ku, Tokyo 171-8501, Japan

^{a)} kakei@rikkyo.ac.jp

Received October 29, 2015; Revised May 11, 2016; Published July 1, 2016

Abstract: Tracy and Widom showed that the level spacing function of the Gaussian unitary ensemble is related to a particular solution of the fourth Painlevé equation. We reconsider this problem from the viewpoint of Hirota's bilinear method in soliton theory and present another proof. We also consider the asymptotic behavior of the level spacing function as $s \rightarrow \infty$, and its relation to the "Clarkson-McLeod solution" to the Painlevé IV equation.

Key Words: Gaussian unitary ensemble, Painlevé IV, nonlinear Schrödinger equation, similarity reduction, Hirota bilinear method, two-directional Wronskian

1. Introduction

Random matrix theory has its origins in mathematical statistics and nuclear physics [1], and has been applied to many fields such as wireless communications [2]. In the paper [3], Tracy and Widom introduced a limiting distribution function associated with the largest eigenvalue of the Gaussian unitary ensemble (GUE). The Tracy-Widom distribution has been applied to many branches of mathematics such as combinatorics and statistics. (For review, see [4].) Furthermore, recent experimental studies reveal that the Tracy-Widom distribution appears in various physical systems such as slow combustion of paper [5], turbulent liquid crystals [6], fiber laser systems [7], evaporating drops of colloidal suspensions [8]. (For a survey of recent experimental studies, see [9].)

The GUE-Tracy-Widom distribution function $F_2(s)$ is given by

$$F_2(s) = \exp \left[- \int_s^\infty (x-s) q^2(x) dx \right], \quad (1)$$

where $q(x)$ is the unique solution to a special case of the Painlevé II equation ("Hastings-McLeod solution" [10])

$$\frac{d^2 q}{ds^2} = s \frac{dq}{ds} + 2q^3 \quad (2)$$

satisfying the boundary condition $q(s) \sim \text{Ai}(s)$ as $s \rightarrow \infty$. The function $F_2(s)$ is a scaling limit of the distribution function of the largest eigenvalue of the GUE. For the ensemble of $N \times N$ random Hermitian matrices, joint distribution of eigenvalues is given by [1]

$$\mathbb{P}_{2,N}(x_1, x_2, \dots, x_N) = \frac{1}{Z_N} \prod_{1 \leq i < j \leq N} |x_i - x_j|^2 \prod_{i=1}^N e^{-x_i^2}, \quad (3)$$

where the normalization constant $1/Z_N$ is given by

$$\begin{aligned} Z_N &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{1 \leq i < j \leq N} |x_i - x_j|^2 \prod_{i=1}^N e^{-x_i^2} dx_1 \cdots dx_N \\ &= 2^{-N(N-1)/2} \pi^{N/2} \prod_{j=1}^N j!. \end{aligned} \quad (4)$$

We will give a bilinear-theoretical proof of the identity (4) in Appendix A.

The distribution function of the largest eigenvalue $\lambda_{\max}$ is then expressed as

$$F_{2,N}(s) := \mathbb{P}_{2,N}(\lambda_{\max} \leq s) = \int_{-\infty}^s \cdots \int_{-\infty}^s \mathbb{P}_{2,N}(x_1, x_2, \dots, x_N) dx_1 \cdots dx_N, \quad (5)$$

and the Tracy-Widom distribution function $F_2(s)$ is obtained as a scaling limit of $F_{2,N}(s)$ [3, 11]:

$$F_2(s) = \lim_{N \rightarrow \infty} F_{2,N} \left(\sqrt{2N} + \frac{s}{\sqrt{2N^{1/6}}} \right). \quad (6)$$

As discussed in [12], the distribution function $F_{2,N}(s)$ is related to the fourth Painlevé equation

$$\frac{d^2 w}{dz^2} = \frac{1}{2w} \left(\frac{dw}{dz} \right)^2 + \frac{3}{2} w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}. \quad (7)$$

Theorem 1 (Tracy-Widom [12]). *The distribution function $F_{2,N}(s)$ is given explicitly by*

$$F_{2,N}(s) = \exp \left[- \int_s^\infty \left\{ Nw - \frac{z^2}{2} w - \frac{z}{2} w^2 - \frac{w^3}{8} + \frac{1}{8w} \left(\frac{dw}{dz} \right)^2 \right\} dz \right], \quad (8)$$

where $w = w(z)$ is a solution to the Painlevé IV equation with the parameters $\alpha = 2N - 1$ and $\beta = 0$, satisfying the boundary condition

$$w(s) \sim 0 \quad \text{as } s \rightarrow +\infty. \quad (9)$$

The proof of Theorem 1 in [12] is based on the Fredholm determinantal expression,

$$F_{2,N}(s) := \mathbb{P}_{2,N}(\lambda_{\max} \leq s) = \det [I - K_{2,N}]_{L^2(s, \infty)}. \quad (10)$$

The kernel of the integral operator $K_{2,N}$ is given by

$$K_{2,N}(x, z) = \frac{e^{-(x^2+z^2)/2}}{2^N \sqrt{\pi} (N-1)!} \cdot \frac{H_N(x) H_{N-1}(z) - H_{N-1}(x) H_N(z)}{x - z}, \quad (11)$$

where $H_k(x)$ is the k -th Hermite polynomials.

In what follows, we present an alternative proof based on Hirota's bilinear method in soliton theory [13]. The solutions to many soliton equations can be expressed in terms of determinants with Wronski-type or Gram-type structure, and the bilinear forms of the equations are reduced to algebraic identities of determinants [13–17]. We show that the function $F_{2,N}(s)$ can be obtained from “two-directional” Wronskian solutions [16] to the nonlinear Schrödinger (NLS) equation

$$i \frac{\partial q}{\partial t} = \frac{\partial^2 q}{\partial x^2} + 2|q|^2 q. \quad (12)$$

We also consider the asymptotic behavior of $F_{2,N}(s)$ as $s \rightarrow \pm\infty$, and its relation to the “Clarkson-McLeod solution” [18–20] to the Painlevé IV equation.

This paper is dedicated to the memory of Professor Ryogo Hirota, who introduced his bilinear method that is a powerful tool for obtaining a wide class of exact solutions of nonlinear integrable equations, such as the NLS equation and the Painlevé IV equation.

2. Nonlinear Schrödinger equation and the Painlevé IV

In this section, we review Hirota bilinear formulation of the NLS Eq. (12), and its similarity reduction to the Painlevé IV Eq. (7).

The relation between NLS-type equations and the Painlevé IV (7) has been discussed from various perspectives [21–28]. Among these works, our approach is based mainly on the work of Jimbo and Miwa [24], in which they introduced the “ τ -function” in a systematic manner.

Hereafter we will forget the complex structure and consider the following coupled system of nonlinear partial differential equations:

$$-\frac{\partial q}{\partial t_2} + \frac{\partial^2 q}{\partial t_1^2} + 2q^2 r = 0, \quad \frac{\partial r}{\partial t_2} + \frac{\partial^2 r}{\partial t_1^2} + 2qr^2 = 0. \quad (13)$$

Setting $r = \bar{q}$, $t_1 = x$, $t_2 = it$ recovers the NLS Eq. (12). It is well-known that the NLS equation has infinite number of conserved currents [23]. The first three of the conserved currents are given as

$$\mathcal{J}_1 = qr, \quad (14)$$

$$\mathcal{J}_2 = \frac{\partial q}{\partial t_1} r - q \frac{\partial r}{\partial t_1}, \quad (15)$$

$$\mathcal{J}_3 = \frac{\partial^2 q}{\partial t_1^2} r + q \frac{\partial^2 r}{\partial t_1^2} - 2 \frac{\partial q}{\partial t_1} \frac{\partial r}{\partial t_1} + 2q^2 r^2, \quad (16)$$

which satisfy the continuity equations

$$\frac{\partial \mathcal{J}_1}{\partial t_2} = \frac{\partial \mathcal{J}_2}{\partial t_1}, \quad \frac{\partial \mathcal{J}_2}{\partial t_2} = \frac{\partial \mathcal{J}_3}{\partial t_1}. \quad (17)$$

These equations guarantee the existence of “ τ -function” of the NLS Eq. (13) that satisfies [24]

$$\frac{\partial^2}{\partial t_1^2} \log \tau = \mathcal{J}_1, \quad \frac{\partial^2}{\partial t_1 \partial t_2} \log \tau = \mathcal{J}_2, \quad \frac{\partial^2}{\partial t_2^2} \log \tau = \mathcal{J}_3. \quad (18)$$

We introduce $\rho_1 = \rho_1(t_1, t_2)$ and $\rho_2 = \rho_2(t_1, t_2)$ as

$$\rho_1(t_1, t_2) = \tau(t_1, t_2) q(t_1, t_2), \quad \rho_2(t_1, t_2) = \tau(t_1, t_2) r(t_1, t_2), \quad (19)$$

which is the same dependent-variable transformation used in [13, 16] to bilinearize (13). Then one can rewrite the Eqs. (13) and (18) as bilinear differential equations of Hirota-type:

$$(D_1^2 + D_2) \tau \cdot \rho_1 = 0, \quad (20)$$

$$(D_1^2 + D_2) \rho_2 \cdot \tau = 0, \quad (21)$$

$$D_1^2 \tau \cdot \tau = 2\rho_1 \rho_2, \quad (22)$$

$$D_1 D_2 \tau \cdot \tau = 2D_1 \rho_1 \cdot \rho_2, \quad (23)$$

$$D_2^2 \tau \cdot \tau = 2D_1^2 \rho_1 \cdot \rho_2, \quad (24)$$

where we have used the Hirota differential operators D_1, D_2 defined by

$$D_1^m D_2^n f(t_1, t_2) \cdot g(t_1, t_2) = \left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t'_1} \right)^m \left(\frac{\partial}{\partial t_2} - \frac{\partial}{\partial t'_2} \right)^n f(t_1, t_2) g(t'_1, t'_2) \Big|_{\substack{t'_1 = t_1 \\ t'_2 = t_2}}. \quad (25)$$

Straightforward calculation shows

Proposition 2. *Let $\tau(t_1, t_2)$, $\rho_1(t_1, t_2)$, $\rho_2(t_1, t_2)$ be a solution to the bilinear Eqs. (20)–(24). Let λ , c_1 , c_2 be constants and assume $\lambda \neq 0$. Define $\tilde{\tau}(t_1, t_2)$, $\tilde{\rho}_1(t_1, t_2)$, $\tilde{\rho}_2(t_1, t_2)$ as*

$$\tilde{\tau}(t_1, t_2) = \lambda^{-1+(c_1+c_2)/2} \tau(\lambda t_1, \lambda^2 t_2), \quad (26)$$

$$\tilde{\rho}_1(t_1, t_2) = \lambda^{c_1} \rho_1(\lambda t_1, \lambda^2 t_2), \quad (27)$$

$$\tilde{\rho}_2(t_1, t_2) = \lambda^{c_2} \rho_2(\lambda t_1, \lambda^2 t_2). \quad (28)$$

Then $\tilde{\tau}(t_1, t_2)$, $\tilde{\rho}_1(t_1, t_2)$, $\tilde{\rho}_2(t_1, t_2)$ also solve the Eqs. (20)–(24).

Motivated from Proposition 2, we now impose similarity conditions on $\tau(t_1, t_2)$, $\rho_1(t_1, t_2)$, $\rho_2(t_1, t_2)$:

$$\tau(t_1, t_2) = \lambda^{-1+(c_1+c_2)/2} \tau(\lambda t_1, \lambda^2 t_2), \quad (29)$$

$$\rho_1(t_1, t_2) = \lambda^{c_1} \rho_1(\lambda t_1, \lambda^2 t_2), \quad (30)$$

$$\rho_2(t_1, t_2) = \lambda^{c_2} \rho_2(\lambda t_1, \lambda^2 t_2), \quad (31)$$

for all $\lambda \in \mathbb{C}^*$, where c_1 and c_2 are constants. Then $q(t_1, t_2)$ and $r(t_1, t_2)$ satisfy

$$q(t_1, t_2) = \lambda^{1+(c_1-c_2)/2} q(\lambda t_1, \lambda^2 t_2), \quad (32)$$

$$r(t_1, t_2) = \lambda^{1+(-c_1+c_2)/2} r(\lambda t_1, \lambda^2 t_2). \quad (33)$$

Proposition 3 (cf. [24]). *Suppose $\tau(t_1, t_2)$, $q(t_1, t_2)$ and $r(t_1, t_2)$ satisfy the differential Eqs. (13), (18), and the similarity conditions (29), (32), (33). Then one can introduce $F(s)$ as*

$$\tau(t_1, t_2) = (2\sqrt{-t_2})^{(2-c_1-c_2)/2} F(s), \quad s = \frac{-t_1}{2\sqrt{-t_2}}. \quad (34)$$

Furthermore, if we define $H(s)$ as

$$H(s) = \frac{d}{ds} \log F(s) = \frac{F'(s)}{F(s)}, \quad (35)$$

then $H(s)$ solves the following differential equation,

$$(H'')^2 - 4(H - sH')^2 + 4H' \{(H')^2 - (c_1 - c_2)H' + 2(c_1 + c_2 - 2)\} = 0. \quad (36)$$

Proof. Setting $\lambda = 1/(2\sqrt{-t_2})$ and $F(s) = \tau(-s, -1/4)$ in (29), we obtain (34). Similarly we obtain

$$\begin{aligned} q(t_1, t_2) &= (2\sqrt{-t_2})^{(-2-c_1+c_2)/2} u(s), \\ r(t_1, t_2) &= (2\sqrt{-t_2})^{(-2+c_1-c_2)/2} v(s) \end{aligned} \quad (37)$$

by setting $u(s) = q(-s, -1/4)$ and $v(s) = r(-s, -1/4)$. Substituting (37) for (13), we have

$$\begin{aligned} u'' &= 2su' + 2u^2v + (c_1 - c_2 + 2)u, \\ v'' &= -2sv' + 2uv^2 + (c_1 - c_2 - 2)v. \end{aligned} \quad (38)$$

In the same manner, substituting (34) and (37) for (18), we obtain

$$\begin{aligned} H' + uv &= 0, \quad sH' + H + \frac{1}{2}(u'v - uv') = 0, \\ 4s^2H' + 12sH + u''v - 2u'v' + uv'' - 2u^2v^2 + 4(c_1 + c_2 - 2) &= 0. \end{aligned} \quad (39)$$

Eliminating u, v from the Eq. (39), one obtains (36). $\square$

Denote as $2(\theta_\infty \pm \theta_0)$ the roots of a quadratic polynomial $x^2 - (c_1 - c_2)x + 2(c_1 + c_2 - 2)$. Then (36) is written as

$$(H'')^2 - 4(sH' - H)^2 + 4H'(H' - 2\theta_\infty - 2\theta_0)(H' - 2\theta_\infty + 2\theta_0) = 0. \quad (40)$$

Setting $\sigma = H - 2t(\theta_\infty + \theta_0)$, we obtain the σ -form of the Painlevé IV equation [24]:

$$(\sigma'')^2 - 4(s\sigma' - \sigma)^2 + 4\sigma'(\sigma' + 4\theta_0)(\sigma' + 2\theta_\infty + 2\theta_0) = 0 \quad (41)$$

(cf. [27, 29–31]).

Proposition 4 ([24, 27]). *Define $Q = Q(s)$, $P = P(s)$ as*

$$Q(s) = -\frac{\sigma'(s)}{2} = -\frac{H'(s)}{2} + \theta_\infty + \theta_0, \quad (42)$$

$$P(s) = \frac{-\sigma''(s) - 2\{s\sigma'(s) - \sigma(s)\}}{2\{\sigma'(s) + 2\theta_\infty + 2\theta_0\}} = \frac{-H''(s) - 2\{sH'(s) - H(s)\}}{2H'(s)}. \quad (43)$$

Then H can be expressed in terms of Q and P :

$$H = \frac{2}{P}Q^2 - \left(P + 2s + \frac{4\theta_0}{P}\right)Q + (\theta_0 + \theta_\infty)(P + 2s), \quad (44)$$

and we have the Hamiltonian system associated with the Painlevé IV:

$$\frac{dQ}{ds} = P \frac{\partial H}{\partial P} = -\frac{2}{P}Q^2 + \left(-P + \frac{4\theta_0}{P}\right)Q + (\theta_0 + \theta_\infty)P, \quad (45)$$

$$\frac{dP}{ds} = -P \frac{\partial H}{\partial Q} = -4Q + P^2 + 2sP + 4\theta_0. \quad (46)$$

Proof. Straightforward calculation shows that P of (43) satisfies

$$H'P^2 + H''P - (H' - 2\theta_0 - 2\theta_\infty)(H' + 2\theta_0 - 2\theta_\infty) = 0. \quad (47)$$

The differential equation

$$\frac{dQ}{ds} = -\frac{2}{P}Q^2 + \left(-P + \frac{4\theta_0}{P}\right)Q + (\theta_0 + \theta_\infty)P \quad (48)$$

follows from (47). The relation (44) follows from (42), (43) and (48). Differentiating (44), one obtains (46). $\square$

The following proposition can be verified by straightforward calculation.

Proposition 5. *Suppose $H = H(s)$ satisfies (40) and define $P(s)$ as (43). Then $w = P(z)$ satisfies the Painlevé IV (7) with $\alpha = 2\theta_\infty - 1$, $\beta = -8\theta_0^2$.*

We remark that the Hamiltonian (44) can be written in terms of $w(s) = P(s)$:

$$H(s) = \theta_\infty w - \frac{s^2}{2}w - \frac{s}{2}w^2 - \frac{1}{8}w^3 + \frac{(w')^2}{8w} - \frac{2\theta_0^2}{w} + 2\theta_\infty s. \quad (49)$$

3. Two-directional Wronskian and GUE

We start considering a class of solution of the bilinear Eqs. (20)–(24), which is expressed in terms of “two-directional” Wronski determinant:

$$\tau_N(t_1, t_2) = \det \left[\partial_1^{i+j-2} f(t_1, t_2) \right]_{i,j=1}^N. \quad (50)$$

Hereafter we use the abbreviation $\partial_1 = \partial/\partial t_1$, $\partial_2 = \partial/\partial t_2$.

Theorem 6 (cf. [16]). *Suppose the function $f(t_1, t_2)$ satisfies*

$$\partial_2 f(t_1, t_2) = \partial_1^2 f(t_1, t_2). \quad (51)$$

Then the τ -function (50) gives a class of solutions for the bilinear Eqs. (20)–(24) with $\tau = \tau_N(t_1, t_2)$, $\rho_1 = \tau_{N+1}(t_1, t_2)$, $\rho_2 = \tau_{N-1}(t_1, t_2)$.

The Eqs. (20), (21), (22) are reduced cases of the Eqs. (4.10a), (4.10b), (4.10c) of [16], respectively. One can find a proof in §4 of [16] and we omit here. We shall give a proof for the remaining two equations in Appendix B.

Here we choose $f(t_1, t_2)$ as

$$f(t_1, t_2) = \int_{-\infty}^0 e^{kt_1 + k^2 t_2} dk. \quad (52)$$

This function admits self-similarity as

$$f(\lambda t_1, \lambda^2 t_2) = \int_{-\infty}^0 e^{k\lambda t_1 + k^2 \lambda^2 t_2} dk = \int_{-\infty}^0 e^{kt_1 + k^2 t_2} \frac{dk}{\lambda} = \lambda^{-1} f(t_1, t_2) \quad (53)$$

for $\lambda > 0$, and hence the two-directional Wronskian (50) also has self-similarity:

$$\tau_N(\lambda t_1, \lambda^2 t_2) = \lambda^{-N^2} \tau_N(t_1, t_2). \quad (54)$$

Thus the scaling exponent for $\rho_1(t_1, t_2) = \tau_{N+1}(t_1, t_2)$ is $c_1 = (N+1)^2$, and that for $\rho_2(t_1, t_2) = \tau_{N-1}(t_1, t_2)$ is $c_2 = (N-1)^2$. It follows from Proposition 3 that

$$F(s) = (2\sqrt{-t_2})^{-(2-c_1-c_2)/2} \tau_N(t_1, t_2) = (2\sqrt{-t_2})^{N^2} \tau_N(t_1, t_2) \quad (55)$$

depends only on $s = -t_1/(2\sqrt{-t_2})$. Since $c_1 = (N+1)^2$, $c_2 = (N-1)^2$, $H(s)$ of (35) satisfies the σ -form Painlevé IV (40) with $\theta_0 = 0$, $\theta_\infty = N$:

$$(H'')^2 - 4(H - sH')^2 + 4H'(H' - 2N)^2 = 0. \quad (56)$$

The two-directional Wronskian τ_N under consideration coincides with the distribution function $F_{2,N}(s)$ as shown below. For this purpose, we assume $t_2 < 0$ and substitute $f(t_1, t_2)$ of (52) into (50):

$$\begin{aligned} \tau_N &= \det \left[\int_{-\infty}^0 k^{i+j-2} e^{kt_1+k^2 t_2} dk \right]_{i,j=1}^N \\ &= \frac{e^{Ns^2}}{(-t_2)^{N^2/2}} \det \left[\int_{-\infty}^s (k-s)^{i+j-2} e^{-k^2} dk \right]_{i,j=1}^N \\ &= \frac{e^{Ns^2}}{(-t_2)^{N^2/2}} \det \left[\int_{-\infty}^s k^{i+j-2} e^{-k^2} dk \right]_{i,j=1}^N. \end{aligned} \quad (57)$$

On the other hand, applying an integral identity [32–34]

$$\begin{aligned} &\int \cdots \int \det [\phi_j(x_k)]_{j,k=1,\dots,N} \det [\psi_j(x_k)]_{j,k=1,\dots,N} dx_1 \cdots x_N \\ &= N! \det \left[\int \phi_j(x) \psi_k(x) dx \right]_{j,k=1,\dots,N} \end{aligned} \quad (58)$$

to (5), one can rewrite $F_{2,N}(s)$ as

$$F_{2,N}(s) = \frac{2^{N(N-1)/2} \pi^{-N/2}}{\prod_{j=1}^{N-1} j!} \det \left[\int_{-\infty}^s x^{j+k-2} e^{-x^2} dx \right]_{j,k=1,\dots,N}. \quad (59)$$

Comparing (55), (57), and (59), we obtain

$$F(s) = 2^{N(N+1)/2} \pi^{N/2} e^{Ns^2} \left\{ \prod_{j=1}^{N-1} (j!) \right\} F_{2,N}(s), \quad (60)$$

and thus

$$H(s) = 2Ns + R(s), \quad R(s) := \frac{d}{ds} \log F_{2,N}(s). \quad (61)$$

It follows from (49) and (61) that

$$\frac{d}{ds} \log F_{2,N}(s) = Nw - \frac{s^2}{2}w - \frac{s}{2}w^2 - \frac{w^3}{8} + \frac{(w')^2}{8w}. \quad (62)$$

Integrating once, we obtain (8).

4. Asymptotic behavior

Real solutions to the Painlevé IV equation in the case $\beta = 0$ and real α with the boundary condition (9) are called “Clarkson-McLeod solution” [18–20].

Theorem 7 (Clarkson-McLeod solution [18, 19] (cf. [20, 35, 36])). *In the case $\beta = 0$ and real α , there exists a unique one-parameter family $w(z; k^2)$ ($k^2 \in \mathbb{R}$) of real solutions of the Painlevé IV Eq. (7) determined by asymptotic condition*

$$w(s) \sim k^2 2^{(\alpha+2)/2} s^{\alpha-1} e^{-s^2} \quad \text{as } s \rightarrow +\infty. \quad (63)$$

Moreover, in the case $\beta = 0$ and real α , any solutions of the Painlevé IV Eq. (7) with the boundary condition (9) coincides with $w(x; k^2)$ for some constant $k^2 \in \mathbb{R}$.

The parameters k, α for the solution of the Painlevé IV associated with $F_{2,N}(s)$ have been evaluated by Tracy and Widom ([12], p.55). We shall consider two integrals

$$I_n := \int_{-\infty}^{\infty} x^n e^{-x^2} dx, \quad J_n := \int_s^{\infty} x^n e^{-x^2} dx \quad (n = 0, 1, 2, \dots) \quad (64)$$

and define $T_N(s; \mu)$ as

$$T_N(s; \mu) := \det [I_{i+j-2} - \mu J_{i+j-2}]_{i,j=1}^N. \quad (65)$$

In the case $\mu = 1$, $T_N(s; 1)$ coincides with the determinant appeared in (59):

$$T_N(s; 1) = \det [I_{i+j-2} - J_{i+j-2}]_{i,j=1}^N = \det \left[\int_{-\infty}^s x^{i+j-2} e^{-x^2} dx \right]_{i,j=1}^N. \quad (66)$$

If we set

$$f(t_1, t_2) = \int_{-\infty}^{\infty} x^n e^{-x^2} dx - \mu \int_0^{\infty} x^n e^{-x^2} dx \quad (67)$$

instead of (52), the corresponding τ -function (50) is related to $T_N(s; \mu)$ as

$$\tau_N = \frac{e^{Ns^2}}{(-t_2)^{N^2/2}} T_N(s; \mu). \quad (68)$$

It follows from Proposition 3 that

$$R(s; \mu) := \frac{d}{ds} \log T_N(s; \mu) \quad (69)$$

satisfies the differential equation

$$(R'')^2 - 4(R - sR')^2 + 4(R')^2(R' + 2N) = 0. \quad (70)$$

The function $R(s)$ of (61) is a special case of (69): $R(s) = R(s; \mu = 1)$.

Theorem 8 (Tracy-Widom [12]). $R(s; \mu) = \{\log T_N(s; \mu)\}'$ is given explicitly by

$$R(s; \mu) = Nw - \frac{s^2}{2}w - \frac{s}{2}w^2 - \frac{w^3}{8} + \frac{(w')^2}{8w}, \quad (71)$$

where $w = w(s; k^2)$ is the Clarkson-McLeod solution (Theorem 7) to the Painlevé IV, with the parameters $\alpha = 2N - 1$, $\beta = 0$, and $k^2 = 2^{-3/2}\pi^{-1/2}\mu/(N-1)!$.

Since no explicit proof of Theorem 8 has been written in [12], we give a proof based on the determinant expression (59).

We remark that J_n can be represented in terms of the incomplete gamma function,

$$\Gamma(a, z) = \int_z^{\infty} t^{a-1} e^{-t} dt, \quad (72)$$

as

$$J_n = \frac{1}{2} \Gamma((n+1)/2, s^2). \quad (73)$$

Using the asymptotic series [37]

$$\Gamma(a, z) \sim z^{a-1} e^{-z} \left[1 + \frac{a-1}{z} + \dots \right] \quad \left(\text{as } z \rightarrow \infty \text{ in } |\arg z| < \frac{3\pi}{2} \right), \quad (74)$$

we can estimate asymptotic behavior of $J_n(s)$ as

$$J_n \sim \frac{1}{2} s^{n-1} e^{-s^2} \left[1 + \frac{n-1}{2s^2} + \dots \right] \quad (\text{as } s \rightarrow \infty). \quad (75)$$

Applying (75) to (65), we have

$$T_N(s; \mu) \sim \det [I_{i+j-2}]_{i,j=1}^N - \frac{\mu}{2} \det [I_{i+j-2}]_{i,j=1}^{N-1} s^{2N-3} e^{-s^2} \quad (\text{as } s \rightarrow \infty). \quad (76)$$

It follows from (4) and (58) that

$$\det [I_{i+j-2}]_{i,j=1}^N = \frac{1}{N!} Z_N = 2^{-N(N-1)/2} \pi^{N/2} \prod_{j=1}^{N-1} j!, \quad (77)$$

and thus

$$T_N(s; \mu) \sim \frac{Z_N}{N!} - \frac{\mu}{2} \frac{Z_{N-1}}{(N-1)!} s^{2N-3} e^{-s^2} \quad (\text{as } s \rightarrow \infty). \quad (78)$$

Applying (78) to (69), we obtain

$$\begin{aligned} R(s; \mu) &\sim \mu N \frac{Z_{N-1}}{Z_N} s^{2N-2} e^{-s^2} \quad (\text{as } s \rightarrow \infty) \\ &= \frac{2^{N-1} \mu}{\pi^{1/2} (N-1)!} s^{2N-2} e^{-s^2}. \end{aligned} \quad (79)$$

Proposition 9. *Higher order expansion of $R(s; \mu)$ is given as*

$$\begin{aligned} R(s; \mu) &\sim \frac{2^{N-1} \mu}{\pi^{1/2} (N-1)!} s^{2N-2} e^{-s^2} \\ &\times \left[1 - \frac{(N-1)(N-3)}{2s^2} + \frac{(N-1)(N-2)(N^2 - 7N + 15)}{8s^4} + \dots \right] \quad (\text{as } s \rightarrow \infty). \end{aligned} \quad (80)$$

Proof. Consider the asymptotic expansion

$$R(s; \mu) \sim s^{2N-2} e^{-s^2} (c_0 + c_1 s^{-1} + c_2 s^{-2} + c_3 s^{-3} + \dots) \quad (81)$$

with $c_0 = 2^{N-1} \pi^{-1/2} \mu / (N-1)!$. Inserting (81) into (70), one can obtain $c_1, c_2, c_3, \dots$ recursively, and the desirous result follows. $\square$

Proof of Theorem 8. From (43) and (61), we have

$$w(s) = \frac{-R'' - 2sR' + 2R}{2(R' + 2N)}. \quad (82)$$

Inserting the asymptotic series (80) to (82), we have the desirous result. $\square$

So far we have considered the asymptotics as $s \rightarrow +\infty$. One can evaluate the other asymptotics as $s \rightarrow -\infty$ of $T_N(s; \mu)$. We now define $\tilde{J}_n(s)$ by

$$\tilde{J}_n(s) := I_n - J_n(s) = \int_{-\infty}^s x^n e^{-x^2} dx, \quad (83)$$

which has the following asymptotics:

$$\tilde{J}_n(s) = -\frac{1}{2} s^{n-1} e^{-s^2} \left[1 + \frac{n-1}{2s^2} + \dots \right] \quad (\text{as } s \rightarrow -\infty). \quad (84)$$

Proposition 10. As $s \rightarrow -\infty$, $R(s; \mu)$ has the asymptotic property

$$R(s; \mu) \sim \begin{cases} -\frac{2^{N-1}}{\pi^{1/2}(N-1)!} \cdot \frac{\mu}{\mu-1} s^{2N-2} e^{-s^2} & (\mu \neq 1), \\ -2Ns & (\mu = 1). \end{cases} \quad (85)$$

Proof. In the case $\mu \neq 1$, we can rewrite $T_N(s; \mu)$ as

$$T_N(s; \mu) = (1 - \mu)^N \tilde{T}_N(s; \tilde{\mu}), \quad (86)$$

with

$$\tilde{T}_N(s; \tilde{\mu}) := \det \left[I_{i+j-2} - \tilde{\mu} \tilde{J}_{i+j-2}(s) \right]_{i,j=1}^N, \quad \tilde{\mu} := \frac{\mu}{\mu-1}. \quad (87)$$

Using (84), one can evaluate the asymptotics as $s \rightarrow -\infty$:

$$\tilde{T}_N(s; \tilde{\mu}) \sim \frac{Z_N}{N!} + \frac{\tilde{\mu}}{2} \frac{Z_{N-1}}{(N-1)!} s^{2N-3} e^{-s^2} \quad (\text{as } s \rightarrow -\infty). \quad (88)$$

Thus we have

$$R(s; \mu) = \frac{d}{ds} \log \tilde{T}_N(s; \tilde{\mu}) \sim -\frac{2^{N-1} \tilde{\mu}}{\pi^{1/2}(N-1)!} s^{2N-2} e^{-s^2} \quad (\text{as } s \rightarrow -\infty). \quad (89)$$

In the case $\mu = 1$, it follows that

$$\begin{aligned} T_N(s; 1) &= \det \left[\tilde{J}_{i+j-2}(s) \right]_{i,j=1}^N \\ &\sim \frac{(-s)^{N(N-2)}}{2^N} e^{-Ns^2} \det \left[1 + \frac{i+j-3}{2s^2} + \cdots \right] \quad (\text{as } s \rightarrow -\infty) \\ &\sim \frac{(-s)^{N(N-2)}}{2^N} e^{-Ns^2} \left\{ \frac{\tilde{C}}{s^{2(N-1)}} + \cdots \right\} \end{aligned} \quad (90)$$

where $\tilde{C}$ is a constant, and we obtain $R(s; 1) \sim -2Ns$ as $s \rightarrow -\infty$. $\square$

Higher order corrections can be obtained in the same manner as Proposition 9. We remark that the asymptotic property $R(s; 1) \sim -2Ns$ (as $s \rightarrow -\infty$) has been mentioned in [12] (p.55, (5.17)).

Acknowledgments

The author acknowledges Takashi Imamura, Makoto Katori, and Tomohiro Sasamoto for discussions on the Tracy-Widom distributions. Thanks are also due to Kenji Kajiwara and Ralph Willox for information on bilinear forms of the Painlevé equations. We are indebted to Pierpaolo Vivo for providing insightful comments and providing directions for additional work. We thank the anonymous reviewers for their careful reading of our manuscript and their many insightful comments and suggestions. This work is partially supported by JSPS KAKENHI Grant No. 23540252 and No. 16K05184.

Appendix

A. Bilinear-theoretic derivation of (4)

In [1], the proof of the integral formula (4) is given by using the Selberg integral. Here we give another proof based on the determinant expression

$$Z_N = N! \det [I_{i+j-2}]_{i,j=1}^N, \quad I_n = \int_{-\infty}^{\infty} x^n e^{-x^2} dx, \quad (A-1)$$

which is a consequence of the identity (58). If n is odd then $I_n = 0$, and if n is even, the integral I_n is a special value of the Gamma function:

$$I_{2m} = \int_0^{\infty} t^{(2m-1)/2} e^{-t} dt = \Gamma\left(m + \frac{1}{2}\right) = \sqrt{\pi} \prod_{j=1}^m \left(m - \frac{1}{2}\right). \quad (A-2)$$

We recall the Desnanot-Jacobi identity for determinants [13, 38] (or “Lewis Carroll identity”, “Dodgson condensation”). We consider a determinant D of degree N and its minors. We denote by $D \begin{bmatrix} i \\ j \end{bmatrix}$ the minor determinant obtained by deleting the i th row and the j th column, and by $D \begin{bmatrix} i_1 & i_2 \\ j_1 & j_2 \end{bmatrix}$ the minor obtained by deleting the i_1 th and i_2 th rows and the j_1 th and j_2 th columns.

Theorem 11 (Desnanot-Jacobi identity [13, 38]). *For i_1, i_2, j_1, j_2 that satisfy $1 \leq i_1 < i_2 \leq N$ and $1 \leq j_1 < j_2 \leq N$, the following identity holds:*

$$D \cdot D \begin{bmatrix} i_1 & i_2 \\ j_1 & j_2 \end{bmatrix} = D \begin{bmatrix} i_1 \\ j_1 \end{bmatrix} D \begin{bmatrix} i_2 \\ j_2 \end{bmatrix} - D \begin{bmatrix} i_1 \\ j_2 \end{bmatrix} D \begin{bmatrix} i_2 \\ j_1 \end{bmatrix}. \quad (\text{A-3})$$

For $M = 0, 1, 2, \dots, N = 1, 2, \dots$, define A_N^M as

$$A_N^M = \det [I_{M+i+j-2}]_{i,j=1}^N. \quad (\text{A-4})$$

From the Desnanot-Jacobi identity (A-3), we have

$$A_{N+2}^M A_N^{M+2} = A_{N+1}^M A_{N+1}^{M+2} - (A_{N+1}^{M+1})^2. \quad (\text{A-5})$$

Setting $M = 2m$ and $N = 2n$, we obtain

$$A_{2n+2}^{2m} A_{2n}^{2m+2} = A_{2n+1}^{2m} A_{2n+1}^{2m+2} \quad (\text{A-6})$$

where we have used $A_{2n+1}^{2m+1} = 0$. Similarly we have

$$A_{2n+1}^{2m} A_{2n-1}^{2m+2} = A_{2n}^{2m} A_{2n}^{2m+2} - (A_{2n}^{2m+1})^2, \quad (\text{A-7})$$

$$A_{2n+2}^{2m-1} A_{2n}^{2m+1} = - (A_{2n+1}^{2m})^2, \quad (\text{A-8})$$

$$0 = A_{2n}^{2m-1} A_{2n}^{2m+1} - (A_{2n}^{2m})^2. \quad (\text{A-9})$$

The values of A_N^M ($M, N = 0, 1, 2, \dots$) are determined by the recursion relations (A-6), (A-7), (A-8), (A-9), together with the initial conditions

$$A_0^M = 1, \quad A_1^M = I_M \quad (M = 0, 1, 2, \dots). \quad (\text{A-10})$$

Proposition 12. *Define $P(l; m, n)$ as*

$$P(l; m, n) = \begin{cases} 1 & (m = 0, n = 1, 2, \dots), \\ \prod_{i=1}^m \prod_{j=1}^n (l + i + j - \frac{3}{2}) & (m, n = 1, 2, \dots). \end{cases} \quad (\text{A-11})$$

Then A_N^M is expressed in terms of $P(l; m, n)$ as

$$A_{2n}^{2m} = \pi^n P(0; m, n) P(1; m, n) \prod_{j=1}^{2n-1} \frac{j!}{2}, \quad (\text{A-12})$$

$$A_{2n+1}^{2m} = \pi^{(2n+1)/2} P(0; m, n+1) P(1; m, n) \prod_{j=1}^{2n} \frac{j!}{2}, \quad (\text{A-13})$$

$$A_{2n}^{2m+1} = (-\pi)^n P(1; m, n)^2 \prod_{j=0}^{n-1} \left\{ \frac{(2j+1)!}{2^{2j+1}} \right\}^2, \quad (\text{A-14})$$

$$A_{2n+1}^{2m+1} = 0 \quad (m, n = 0, 1, 2, \dots). \quad (\text{A-15})$$

Proof. It is enough to show that (A-12), (A-13), (A-14) satisfy the recursion relations (A-6), (A-7), (A-8), (A-9), and the initial conditions (A-10), that can be checked by straightforward calculation. $\square$

The formula (4) follows from a special case of Proposition 12:

$$Z_N = N! A_N^0 = N! \cdot \pi^{N/2} \prod_{j=1}^{N-1} \frac{j!}{2} = 2^{-N(N-1)/2} \pi^{N/2} \prod_{j=1}^N j!. \quad (\text{A-16})$$

B. Proof of Theorem 6

In this appendix, we show that the two-directional Wronskian (50) actually satisfies the bilinear Eqs. (23) and (24) with $\tau = \tau_N(t_1, t_2)$, $\rho_1 = \tau_{N+1}(t_1, t_2)$, $\rho_2 = \tau_{N-1}(t_1, t_2)$. We first introduce the following notation, which is a generalization of the abbreviated notation due to Freeman and Nimmo [13, 15–17]:

$$\begin{vmatrix} m_1, m_2, \dots, m_N \\ n_1, n_2, \dots, n_N \end{vmatrix} := \det \left[\partial_1^{m_i + n_j} f(t_1, t_2) \right]_{i,j=1}^N. \quad (\text{B-1})$$

The τ -function (50) and its derivatives are written as

$$\tau_N = \begin{vmatrix} 0, \dots, N-1 \\ 0, \dots, N-1 \end{vmatrix}, \quad (\text{B-2})$$

$$\partial_1 \tau_N = \begin{vmatrix} 0, \dots, N-1 \\ 0, \dots, N-2, N \end{vmatrix} = \begin{vmatrix} 0, \dots, N-2, N \\ 0, \dots, N-1 \end{vmatrix}, \quad (\text{B-3})$$

$$\partial_1^2 \tau_N = \begin{vmatrix} 0, \dots, N-2, N \\ 0, \dots, N-2, N \end{vmatrix} = \begin{vmatrix} 0, \dots, N-3, N-1, N \\ 0, \dots, N-1 \end{vmatrix} + \begin{vmatrix} 0, \dots, N-2, N+1 \\ 0, \dots, N-1 \end{vmatrix}. \quad (\text{B-4})$$

Using (51), we have

$$\partial_2 \tau_N = - \begin{vmatrix} 0, \dots, N-3, N-1, N \\ 0, \dots, N-1 \end{vmatrix} + \begin{vmatrix} 0, \dots, N-2, N+1 \\ 0, \dots, N-1 \end{vmatrix}, \quad (\text{B-5})$$

$$\partial_1 \partial_2 \tau_N = - \begin{vmatrix} 0, \dots, N-3, N-1, N \\ 0, \dots, N-2, N \end{vmatrix} + \begin{vmatrix} 0, \dots, N-2, N+1 \\ 0, \dots, N-2, N \end{vmatrix}, \quad (\text{B-6})$$

$$\begin{aligned} \partial_2^2 \tau_N &= \begin{vmatrix} 0, \dots, N-3, N-1, N \\ 0, \dots, N-3, N-1, N \end{vmatrix} \\ &\quad - 2 \begin{vmatrix} 0, \dots, N-3, N-1, N \\ 0, \dots, N-2, N+1 \end{vmatrix} + \begin{vmatrix} 0, \dots, N-2, N+1 \\ 0, \dots, N-2, N+1 \end{vmatrix}. \end{aligned} \quad (\text{B-7})$$

Using the differential formulas (B-3) and the identity (A-3), we obtain

$$\begin{aligned} (\partial_1 \tau_{N+1}) \tau_{N-1} &= \begin{vmatrix} 0, \dots, N-1, N+1 \\ 0, \dots, N \end{vmatrix} \begin{vmatrix} 0, \dots, N-2 \\ 0, \dots, N-2 \end{vmatrix} \\ &= \begin{vmatrix} 0, \dots, N-2, N+1 \\ 0, \dots, N-2, N \end{vmatrix} \begin{vmatrix} 0, \dots, N-1 \\ 0, \dots, N-1 \end{vmatrix} \\ &\quad - \begin{vmatrix} 0, \dots, N-2, N+1 \\ 0, \dots, N-1 \end{vmatrix} \begin{vmatrix} 0, \dots, N-1 \\ 0, \dots, N-2, N \end{vmatrix}, \end{aligned} \quad (\text{B-8})$$

$$\begin{aligned} \tau_{N+1} (\partial_1 \tau_{N-1}) &= \begin{vmatrix} 0, \dots, N \\ 0, \dots, N \end{vmatrix} \begin{vmatrix} 0, \dots, N-3, N-1 \\ 0, \dots, N-2 \end{vmatrix} \\ &= \begin{vmatrix} 0, \dots, N-3, N-1, N \\ 0, \dots, N-2, N \end{vmatrix} \begin{vmatrix} 0, \dots, N-1 \\ 0, \dots, N-1 \end{vmatrix} \\ &\quad - \begin{vmatrix} 0, \dots, N-3, N-1, N \\ 0, \dots, N-1 \end{vmatrix} \begin{vmatrix} 0, \dots, N-1 \\ 0, \dots, N-2, N \end{vmatrix}. \end{aligned} \quad (\text{B-9})$$

It is straightforward to show that $\tau = \tau_N$, $\rho_1 = \tau_{N+1}$, $\rho_2 = \tau_{N-1}$ solve (23) by using (B-8), (B-9), and the differential relations (B-3), (B-5), (B-6).

We use the Desnanot-Jacobi identity (A-3) again to obtain

$$(\partial_1^2 \tau_{N+1}) \tau_{N-1} = \begin{vmatrix} 0, \dots, N-1, N+1 \\ 0, \dots, N-1, N+1 \end{vmatrix} \begin{vmatrix} 0, \dots, N-2 \\ 0, \dots, N-2 \end{vmatrix}$$

$$\begin{aligned}
&= \begin{vmatrix} 0, \dots, N-2, N+1 \\ 0, \dots, N-2, N+1 \end{vmatrix} \begin{vmatrix} 0, \dots, N-1 \\ 0, \dots, N-1 \end{vmatrix} \\
&\quad - \begin{vmatrix} 0, \dots, N-2, N+1 \\ 0, \dots, N-1 \end{vmatrix} \begin{vmatrix} 0, \dots, N-1 \\ 0, \dots, N-2, N+1 \end{vmatrix}, \tag{B-10}
\end{aligned}$$

$$\begin{aligned}
\tau_{N+1} (\partial_1^2 \tau_{N-1}) &= \begin{vmatrix} 0, \dots, N \\ 0, \dots, N \end{vmatrix} \begin{vmatrix} 0, \dots, N-3, N-1 \\ 0, \dots, N-3, N-1 \end{vmatrix} \\
&= \begin{vmatrix} 0, \dots, N-3, N-1, N \\ 0, \dots, N-3, N-1, N \end{vmatrix} \begin{vmatrix} 0, \dots, N-1 \\ 0, \dots, N-1 \end{vmatrix} \\
&\quad - \begin{vmatrix} 0, \dots, N-3, N-1, N \\ 0, \dots, N-1 \end{vmatrix} \begin{vmatrix} 0, \dots, N-1 \\ 0, \dots, N-3, N-1, N \end{vmatrix}, \tag{B-11}
\end{aligned}$$

$$\begin{aligned}
(\partial_1 \tau_{N+1}) (\partial_1 \tau_{N-1}) &= \begin{vmatrix} 0, \dots, N-1, N+1 \\ 0, \dots, N \end{vmatrix} \begin{vmatrix} 0, \dots, N-2 \\ 0, \dots, N-3, N-1 \end{vmatrix} \\
&= \begin{vmatrix} 0, \dots, N-2, N+1 \\ 0, \dots, N-3, N-1, N \end{vmatrix} \begin{vmatrix} 0, \dots, N-1 \\ 0, \dots, N-1 \end{vmatrix} \\
&\quad - \begin{vmatrix} 0, \dots, N-2, N+1 \\ 0, \dots, N-1 \end{vmatrix} \begin{vmatrix} 0, \dots, N-1 \\ 0, \dots, N-3, N-1, N \end{vmatrix}. \tag{B-12}
\end{aligned}$$

These relations give (24) with $\tau = \tau_N$, $\rho_1 = \tau_{N+1}$, and $\rho_2 = \tau_{N-1}$.

C. Bilinearization of the Painlevé IV

As in Proposition 3, under the similarity conditions (29)–(31), one can introduce $F(s)$ as (34) and $G_1(s)$, $G_2(s)$ as

$$\rho_1(t_1, t_2) = (2\sqrt{-t_2})^{-c_1} G_1(s), \quad \rho_2(t_1, t_2) = (2\sqrt{-t_2})^{-c_2} G_2(s). \tag{C-1}$$

Bilinear equations for $F(s)$, $G_1(s)$ and $G_2(s)$ follow from (20)–(24):

$$(D_s^2 + 2sD_s - c_1 + c_2 - 2) F \cdot G_1 = 0, \tag{C-2}$$

$$(D_s^2 + 2sD_s - c_1 + c_2 + 2) G_2 \cdot F = 0, \tag{C-3}$$

$$D_s^2 F \cdot F = G_1 G_2, \tag{C-4}$$

$$\left(sD_s^2 + \frac{d}{ds} \right) F \cdot F = D_s G_1 \cdot G_2, \tag{C-5}$$

$$\left(2s^2 D_s^2 + 4(c_1 + c_2 - 2) + 6s \frac{d}{ds} \right) F \cdot F = D_s^2 G_1 \cdot G_2. \tag{C-6}$$

These give a bilinearization of the Painlevé IV, however, which are different from that used in [39–41].

References

-
- [1] M.L. Mehta, *Random matrices*, 3rd edition, Elsevier, San Diego, 2004.
 - [2] A.M. Tulino and S. Verdu, *Random matrix theory and wireless communications* “Foundations and Trends in Communications and Information Theory,” Now Publishers, Hanover, 2004.
 - [3] C.A. Tracy and H. Widom, “Level-spacing distributions and the Airy kernel,” *Commun. Math. Phys.*, vol. 159, no. 1, pp. 151–174, January 1994.
 - [4] C.A. Tracy and H. Widom, “Distribution functions for largest eigenvalues and their applications,” in *Proceedings of the International Congress of Mathematicians*, vol. I (Beijing, 2002), pp. 587–596, Higher Education Press, Beijing, 2002.

- [5] L. Miettinen, M. Myllys, J. Merikoski, and J. Timonen, “Experimental determination of KPZ height-fluctuation distributions,” *Eur. Phys. J. B*, vol. 46, no. 1, pp. 55–60, July 2005.
- [6] K.A. Takeuchi and M. Sano, “Universal fluctuations of growing interfaces: evidence in turbulent liquid crystals,” *Phys. Rev. Lett.*, vol. 104, no. 23, 230601, June 2010.
- [7] M. Fridman, R. Pugatch, M. Nixon, A.A. Friesem, and N. Davidson, “Measuring maximal eigenvalue distribution of Wishart random matrices with coupled lasers,” *Phys. Rev. E*, vol. 85, no. 2, 020101(R), February 2012.
- [8] P.J. Yunker, M.A. Lohr, T. Still, A. Borodin, D.J. Durian, and A.G. Yodh, “Effects of particle shape on growth dynamics at edges of evaporating drops of colloidal suspensions,” *Phys. Rev. Lett.*, vol. 110, no. 3, 035501, January 2013.
- [9] K.A. Takeuchi, “Experimental approaches to universal out-of-equilibrium scaling laws: turbulent liquid crystal and other developments,” *J. Stat. Mech.*, vol. 2014, P01006, January 2014.
- [10] S.P. Hastings and J.B. McLeod, “A boundary value problem associated with the second Painlevé transcendent and the Korteweg-de Vries equation,” *Arch. Rat. Mech. Anal.*, vol. 73, no. 1, pp. 31–51, March 1980.
- [11] P.J. Forrester, “The spectrum edge of random matrix ensembles,” *Nuclear Physics B*, vol. 402, no. 3, pp. 709–728, August 1993.
- [12] C.A. Tracy and H. Widom, “Fredholm determinants, differential equations and matrix models,” *Commun. Math. Phys.*, vol. 163, no. 1, pp. 33–72, June 1994.
- [13] R. Hirota, *The direct method in soliton theory*, Cambridge Tracts in Mathematics, vol. 155, Cambridge University Press, Cambridge, 2004.
- [14] J. Satsuma, “A Wronskian representation of N -soliton solutions of nonlinear evolution equations,” *J. Phys. Soc. Jpn.*, vol. 46, no. 1, pp. 359–360, January 1979.
- [15] N.C. Freeman and J.J.C. Nimmo, “Soliton solutions of the Korteweg-de Vries and Kadomtsev-Petviashvili equations: the Wronskian technique,” *Phys. Lett. A*, vol. 95, no. 1, pp. 1–3, April 1983.
- [16] R. Hirota, Y. Ohta, and J. Satsuma, “Wronskian structures of solutions for soliton equations,” *Prog. Theor. Phys. Supplement*, no. 94, pp. 59–72, 1988.
- [17] J.J.C. Nimmo and J. Zhao, “Determinant and Pfaffian solutions of soliton equations,” *Phys. Scr.*, vol. 89, no. 3, 038005, February 2014.
- [18] A.P. Bassom, P.A. Clarkson, A.C. Hicks, and J.B. McLeod, “Integral equations and exact solutions for the fourth Painlevé equation,” *Proc. R. Soc. Lond. A*, vol. 437, no. 1899, pp. 1–24, April 1992.
- [19] P.A. Clarkson and J.B. McLeod, “Integral equations and connection formulae for the Painlevé equations,” in *Painlevé Transcendents: Their Asymptotics and Physical Applications*, eds. P. Winternitz and D. Levi, NATO Adv. Sci. Inst. Ser. B Phys., vol. 278, pp. 1–31, Plenum, New York, 1992.
- [20] A.R. Its and A.A. Kapaev, “Connection formulae for the fourth Painlevé transcendent; Clarkson-McLeod solution,” *J. Phys. A: Math. Gen.*, vol. 31, no. 17, pp. 4073–4113, May 1998.
- [21] R. Nakach, “Self-similar solutions of nonlinear evolution equations of physical significance,” in *Plasma Physics: Nonlinear Theory and Experiments*, ed. H. Wilhelmsson, Nobel Foundation Symposia, vol. 36, pp. 456–474, Springer Science+Business Media, New York, 1977.
- [22] M.J. Ablowitz, A. Ramani, and H. Segur, “A connection between nonlinear evolution equations and ordinary differential equations of P-type. II,” *J. Math. Phys.*, vol. 21, no. 5, pp. 1006–1015, July 1980.
- [23] M.J. Ablowitz and H. Segur, *Solitons and the inverse scattering transform*, SIAM, Philadelphia, 1981.
- [24] M. Jimbo and T. Miwa, “Monodromy preserving deformations of linear ordinary differential equations with rational coefficients II,” *Physica D: Nonlinear Phenomena*, vol. 2, no. 3, pp. 407–448, June 1981.
- [25] A.V. Kitaev, “Self-similar solutions of the modified nonlinear Schrödinger equation,” *Theor. Math. Phys.*, vol. 64, no. 3, pp. 878–894, September 1985.

- [26] M. Can, “On the relation between nonlinear Schrödinger equation and Painlevé IV equation,” *Il Nuovo Cimento B* vol. 106, no. 2, pp. 205–207, February 1991.
- [27] A.P. Bassom, P.A. Clarkson, and A.C. Hicks, “On the application of solutions of the fourth Painlevé equation to various physically motivated nonlinear partial differential equations,” *Adv. Differential Equations*, vol. 1, no. 2, pp. 175–198, January 1996.
- [28] S. Kakei and T. Kikuchi, “Affine Lie group approach to a derivative nonlinear Schrödinger equation and its similarity reduction,” *Int. Math. Res. Notices*, vol. 2004, no. 78, pp. 4181–4209, 2004.
- [29] K. Okamoto, “Studies on the Painlevé equations III. Second and fourth Painlevé equations, P_{II} and P_{IV} ,” *Math. Ann.*, vol. 275, no. 2, pp. 221–255, June 1986.
- [30] P.J. Forrester and N.S. Witte, “Application of the τ -function theory of Painlevé equations to random matrices: PIV, PII and the GUE,” *Commun. Math. Phys.*, vol. 219, no. 2, pp. 357–398, May 2001.
- [31] P.A. Clarkson, “The fourth Painlevé transcendent,” in *Differential Algebra and Related Topics II*, eds. Li Guo and W.Y. Sit, World Scientific, Singapore, 2008. (Also available at <http://kar.kent.ac.uk/id/eprint/23090>)
- [32] C. Andréief, “Note sur une relation les intégrales définies des produits des fonctions,” *Mém. de la Soc. Sci., Bordeaux*, ser. 3, vol. 2, pp. 1–14, 1883.
- [33] N.G. de Bruijn, “On some multiple integrals involving determinants,” *J. Indian Math. Soc.*, vol. 19, pp. 133–151, 1955.
- [34] C.A. Tracy and H. Widom, “Correlation functions, cluster functions and spacing distributions for random matrices,” *J. Stat. Phys.*, vol. 92, no. 5, pp. 809–835, September 1998.
- [35] A.S. Abdullaye, “Justification of asymptotic formulas for the fourth Painlevé equation,” *Stud. Appl. Math.*, vol. 99, no. 3, pp. 255–283, October 1997.
- [36] Y. Lu, “On the asymptotic representation of the solutions to the fourth general Painlevé equation,” *Int. J. Math. Math. Sci.*, vol. 2003, no. 13, pp. 845–851, 2003.
- [37] P.J. Davis, “Gamma function and related functions,” in *Handbook of mathematical functions with formulas: graphs, and mathematical tables*, eds. M. Abramowitz and I.A. Stegun, pp. 255–293, Dover, New York, 1965.
- [38] D.M. Bressoud, *Proofs and confirmations: the story of the alternating sign matrix conjecture*, MAA: Spectrum Series, Cambridge University Press, Cambridge, 1999.
- [39] K. Kajiwara and Y. Ohta, “Determinant structure of the rational solutions for the Painlevé IV equation,” *J. Phys. A: Math. Gen.* vol. 31, no. 10, pp. 2431–2446, March 1998.
- [40] M. Noumi and Y. Yamada, “Symmetries in the fourth Painlevé equation and Okamoto polynomials,” *Nagoya Math. J.*, vol. 153, pp. 53–86, 1999.
- [41] J. Hietarinta and R. Willox, “On the bilinear forms of Painlevé equations,” in *Bilinear Integrable Systems: from Classical to Quantum, Continuous to Discrete*, eds. L.D. Fadeev and P. van Moerbeke, pp. 375–390, Springer, Dordrecht, 2006.