Invited Paper

Algebraic geometric approach to output dead-beat controllability of discrete-time polynomial systems

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Received February 15, 2016; Revised April 22, 2016; Published October 1, 2016

Abstract: In this paper, we consider output dead-beat controllability of discrete-time polynomial systems. First, we derive an output dead-beat stability condition. Based on this stability condition, second, we derive an output dead-beat controllability condition. If our controllability condition holds, it is possible to design an output dead-beat feedback controller in the polynomial form by solving a finite set of algebraic equations.

Key Words: discrete-time systems, polynomial systems, controllability, algebraic geometry

1. Introduction

Various types of controllability are studied for discrete-time nonlinear systems by using various mathematical tools. However, almost papers such as [1–6] study local properties. One of the difficulties of studying global properties are to obtain checkable conditions, i.e., conditions which can be verified by a finite number of computations. Few papers [7–10] study global properties for discrete-time polynomial systems. However, the papers [7, 9, 10] consider specific subclasses of polynomial systems to obtain checkable conditions. The paper [8] considers specific reachability, i.e., reachability in a fixed finite time.

In this paper, we study controllability of general polynomial systems. First, we derive a finite-time output stability condition. Based on this condition, we derive an output dead-beat controllability condition. Moreover, for invertible systems, we derive a reachability condition. Although output dead-beat controllability and reachability conditions are sufficient conditions, it is possible to check them by solving finite sets of algebraic equations.

This paper is motivated by [11], which studies finite-time stability of discrete-time polynomial systems in terms of dimension theory of algebraic geometry. There are several papers utilizing algebraic geometry and commutative algebra for analysis of polynomial systems. For instance, controllability and observability are studied in [6–10] and [12–14], respectively. However, these papers do not employ dimension theory. Owing to dimension theory, it can be shown that the number of compositions of the polynomial mapping when checking finite-time output stability is the same as the dimension of the system. Consequently, the output dead-beat controllability condition is derived in terms of a finite
set of algebraic equations.

The remainder of this paper is organized as follows. In Section 2, basic definitions of commutative algebra and algebraic geometry are summarized. In Section 3, output dead-beat controllability and reachability conditions are presented. Finally, we conclude this paper.

2. Mathematical preliminaries

Here, we summarize basic terms in commutative algebra and algebraic geometry; for more details, see [15–17]. Throughout the paper, \( \mathbb{R} \) denotes the field of real numbers. The set of all polynomials in variables \( z_i, u, \) and \( x_i \) \((i = 1, 2, \ldots, n)\) with coefficients in \( \mathbb{R} \) is denoted by \( \mathbb{R}[z, u, x] := \mathbb{R}[z_1, \ldots, z_n, u, x_1, \ldots, x_n]. \) This \( \mathbb{R}[z, u, x] \) is a commutative ring and thus is called a polynomial ring.

An ideal \( I \) of the polynomial ring \( \mathbb{R}[z, u, x] \) has the following properties for any \( a, b \in I, \) and \( c \in \mathbb{R}[z, u, x]:\)
1. \( a + b \in I, \)
2. \( ca \in I. \)

Let \( p_1, \ldots, p_r \) be polynomials in \( \mathbb{R}[z, u, x]. \) Then, the set \( \langle p_1, \ldots, p_r \rangle, \) defined as
\[
\langle p_1, \ldots, p_r \rangle = \left\{ \sum_{i=1}^{r} a_ip_i : a_1, \ldots, a_r \in \mathbb{R}[z, u, x] \right\},
\]
is an ideal and is called the ideal generated by polynomials \( p_1, \ldots, p_r. \) The set of polynomials \( \{p_1, \ldots, p_r \} \) are called a basis of the ideal. In general, a basis is not unique. However, if we give an order for variables, so called reduced Gröbner basis uniquely exists. The reduced Gröbner basis can be computed by using computer algebra systems such as Maple and Mathematica.

The affine algebraic variety, also called the algebraic set, of polynomials \( p_1, \ldots, p_r \in \mathbb{R}[z, u, x] \) is defined as
\[
V(p_1, \ldots, p_r) = \{ (z_0, u_0, x_0) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : p_i(z_0, u_0, x_0) = 0, i = 1, \ldots, r \}.
\]
An affine variety \( V \subset \mathbb{R}^n \) is said to be irreducible if whenever \( V \) is written in the form \( V = V_1 \cup V_2, \) where \( V_1 \subset \mathbb{R}^n \) and \( V_2 \subset \mathbb{R}^n \) are affine varieties, then either \( V_1 = V \) or \( V_2 = V. \)

In relation to the algebraic variety, the Zariski closure of sets in \( \mathbb{R}^n \) is defined. For a given subset \( A \subset \mathbb{R}^n, \) its Zariski closure, denoted by \( \overline{A}, \) is defined as the smallest algebraic variety containing \( A. \) It is known that the Zariski closure uniquely exists, and \( \overline{A} = \overline{\overline{A}}. \)

3. Output dead-beat controllability condition

3.1 Output dead-beat stability

Let us consider a discrete-time polynomial system. The state equation of the system is described by
\[
\Sigma : \left\{ \begin{array}{l}
x(t + 1) = f(x(t), u(t)), \\
y(t) = h(x(t)),
\end{array} \right.
\]
where \( x(t) \in \mathbb{R}^n, \) \( u(t) \in \mathbb{R}, \) and \( y(t) \in \mathbb{R} \) denote the state, the input, and the output respectively; \( f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) and \( h : \mathbb{R}^n \to \mathbb{R} \) are polynomial mappings where \( f(0, 0) = 0 \) and \( h(0) = 0. \)

Although we consider single-input and single-output systems, our results can directly be generalized to multiple-input and multiple-output systems.

Before defining output dead-beat controllability, we introduce some notations. We denote the composition of function \( f \) as
\[
f_{u_2} \circ f_{u_1}(x) = f(f(x, u_1), u_2).
\]
For the sake of simplicity, \( f_{u_1} \ldots u_1(x) \) denotes \( f_{u_2} \circ \cdots \circ f_{u_1}(x). \) The input sequence \( (u_1, \ldots, u_t) \in \mathbb{R}^t \) is denoted by \( U_t \) \((t \geq 1)\).
Next, consider a feedback controller \( u = k(x) \), where \( k(0) = 0 \). Denote \( f(x, k(x)) \) by \( g(x) \). Note that \( g(0) = 0 \) because \( k(0) = 0 \). We denote the composition of \( g \) as \( g^{i+1}(x) = g \circ g^i(x) \) \((i \geq 0)\), where \( g^0(x) = x \).

**Definition 3.1.** The autonomous system

\[
\begin{align*}
  x(t + 1) &= g(x(t)), \\
  y(t) &= h(x(t)),
\end{align*}
\]  

(1)

is said to be finite-time output stable if there exists a natural number \( N \) such that for all initial states \( x_0 \in \mathbb{R}^n \),

\[ h(g^i(x_0)) = 0 \quad (i \geq N). \]

If \( h \) is the identity mapping on \( \mathbb{R}^n \), the finite-time output stable system is simply called finite-time stable.

**Definition 3.2.** [9] The system \( \Sigma \) is said to be output dead-beat controllable if there exists a natural number \( N \) such that for any initial state \( x_0 \in \mathbb{R}^n \) there exists an input sequence \( U_t = (u_1, \ldots, u_t) \in \mathbb{R}^t \) satisfying

\[ h(f_{u_t}, \ldots, u_1(x_0)) = 0 \quad (t \geq N). \]

If \( h \) is the identity mapping on \( \mathbb{R}^n \), the output dead-beat controllable system is simply called dead-beat controllable.

The original system \( \Sigma \) is dead-beat output controllable if there is a feedback controller which makes the closed-loop system finite-time output stable. Thus, finite-time stability has close relationship with dead-beat controllability. Here, we derive a finite-time output stability condition in terms of algebraic geometry. Since \( \mathbb{R}^n \supset g(\mathbb{R}^n) \) holds, we have \( g(\mathbb{R}^n) \supset g^2(\mathbb{R}^n) \) and thus the following chain

\[ \mathbb{R}^n \supset g(\mathbb{R}^n) \supset g^2(\mathbb{R}^n) \supset \cdots \]

(2)

By taking the Zariski closure, we have

\[ \overline{\mathbb{R}^n} \supset \overline{g(\mathbb{R}^n)} \supset \overline{g^2(\mathbb{R}^n)} \supset \cdots \]

This is a chain of irreducible varieties. Based on dimension theory of algebraic geometry, the following has been shown in [11]:

\[ \overline{\mathbb{R}^n} \supset \overline{g(\mathbb{R}^n)} \supset \cdots \supset \overline{g^n(\mathbb{R}^n)} = \overline{g^{n+1}(\mathbb{R}^n)} = \cdots \]

(3)

Thus, we obtain

\[ h(\overline{\mathbb{R}^n}) \supset h(\overline{g(\mathbb{R}^n)}) \supset \cdots \supset h(\overline{g^n(\mathbb{R}^n)}) = h(\overline{g^{n+1}(\mathbb{R}^n)}) = \cdots, \]

(4)

and consequently

\[ \overline{h(\overline{\mathbb{R}^n})} \supset \overline{h(\overline{g(\mathbb{R}^n)})} \supset \cdots \supset \overline{h(\overline{g^n(\mathbb{R}^n)})} = \overline{h(\overline{g^{n+1}(\mathbb{R}^n)})} = \cdots. \]

From the property of closure, we also have

\[ \overline{h(\overline{\mathbb{R}^n})} \supset \overline{h(\overline{g(\mathbb{R}^n)})} \supset \cdots \supset \overline{h(\overline{g^n(\mathbb{R}^n)})} = \overline{h(\overline{g^{n+1}(\mathbb{R}^n)})} = \cdots. \]

(5)

Based on this equality, we have the following result.

**Theorem 3.1.** The autonomous system (1) is finite-time output stable if and only if \( h(g^n(\mathbb{R}^n)) = \{0\} \).
Proof. (Necessity) If the system is finite-time output stable, there exists \( N \) such that \( h(g^i(\mathbb{R}^n)) = \{0\} \) for all \( i \geq N \), which implies

\[
\overline{h(g^N(\mathbb{R}^n))} = \overline{h(g^{N+1}(\mathbb{R}^n))} = \cdots = \{0\}.
\]

From (5), we have \( h(g^n(\mathbb{R}^n)) = \{0\} \). Since \( \{0\} \) is an affine variety, and since \( h(g^n(\mathbb{R}^n)) \supset \{0\} \), we have \( h(g^n(\mathbb{R}^n)) = \{0\} \).

(Sufficiency) Note that, from (2)

\[
g^n(\mathbb{R}^n) \supset g^{n+1}(\mathbb{R}^n) \supset \cdots
\]

holds. Thus, if \( h(g^n(\mathbb{R}^n)) = \{0\} \) then

\[
h(g^n(\mathbb{R}^n)) = h(g^{n+1}(\mathbb{R}^n)) = \cdots = \{0\}
\]

holds.

The condition in the above theorem can easily be verified. First, polynomial \( h(g^n(x)) \in \mathbb{R}[x] \) is easily computed. Since \( \mathbb{R} \) is an infinite field, \( h(g^n(\mathbb{R}^n)) = \{0\} \subset \mathbb{R} \) holds if and only if \( h(g^n(x)) \) is the zero polynomial \([15]\).

This theorem based on (3) is a natural extension of the results for the linear systems. Extensions of (3) to more general nonlinear mappings are not obvious. In (3), \( g^n(\mathbb{R}^n) = g^{n+1}(\mathbb{R}^n) \) is obtained by taking the Zariski closure, i.e. the closure in the Zariski topology. Since the Zariski topology is a topology on algebraic varieties, \( \overline{g^n(\mathbb{R}^n)} = \overline{g^{n+1}(\mathbb{R}^n)} \) is a specific property of polynomial mappings. On the other hand, even for polynomial mappings, it is unclear whether descending chain (2) stabilizes in a finite time. Therefore, taking the Zariski closure is one of the crucial ideas in Theorem 3.1.

### 3.2 Output dead-beat controllability

Based on Theorem 3.1, we consider to design a polynomial state feedback controller \( u = k(x) \) with a given total degree \( p \geq 1 \):

\[
k(x) = \sum_{|\alpha|=1}^{p} a_\alpha x^\alpha, \; a_\alpha \in \mathbb{R}
\]

where \( |\alpha| = \alpha_1 + \cdots + \alpha_n \) and \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) for \( \alpha_i \geq 0 \) (\( i = 1, \ldots, n \)). Also, denote \( f(x, k(x)) \) by \( g(x) \). Note that \( g(0) = 0 \) because \( k(0) = 0 \). At present, there is no method for finding an upper bound for \( p \). Here, our objective is to find suitable coefficients \( a_\alpha \) for a given \( p \). These \( a_\alpha \) can be obtained by solving a finite set of algebraic equations owing to the following theorem, which follows from Theorem 3.1.

**Theorem 3.2.** For the system \( \Sigma \), there exists a polynomial output dead-beat controller with total degree \( p \geq 1 \) if and only if there exist \( a_\alpha \) such that \( h(g^n(x)) = 0 \).

**Example 3.1.** Consider

\[
\begin{align*}
x_1(t+1) &= x_1(t) + x_2^2(t) + u(t), \\
x_2(t+1) &= x_1(t)(x_1(t) + 2x_2(t)) + x_2(t), \\
y(t) &= x_1(t) + x_2(t).
\end{align*}
\]

For instance, we consider a polynomial state feedback controller \( u \) with the total degree 2:

\[
u = a_{10}x_1 + a_{01}x_2 + a_{20}x_1^2 + a_{11}x_1x_2 + a_{02}x_2^2.
\]

For the closed-loop system, \( h(g^2(x)) \) is obtained as

\[
h(g^2(x)) = x_1 + x_2^2 + x_1(x_1 + 2x_2) + x_2 \\
+ a_{10}x_1 + a_{01}x_2 + a_{20}x_1^2 + a_{11}x_1x_2 + a_{02}x_2^2.
\]
By solving \( h(g^2(x)) = 0 \) with respect to \( a_{i,j} \), we have

\[
a_{10} = -1, \quad a_{01} = -1, \quad a_{20} = -1, \quad a_{02} = -1, \quad a_{11} = -2.
\]

Thus, the system is output dead-beat controllable, and the feedback controller achieving output dead-beat control is

\[
u = -x_1 - x_2 - x_1^2 - 2x_1x_2 - x_2^2.
\]

\[
= -(x_1 + x_2) - (x_1 + x_2)^2.
\]

Now, we notice that \( h(x) = x_1 + x_2 \), and thus the controller can be constructed as an output feedback controller. In general, whether or not a feedback controller can be described as an output feedback controller is not easily verified. However, by using a Gröbner basis, this problem can be solved. For a given controller \( u = \sum_{|\alpha| = 1} a_{\alpha} x^{\alpha} \) and the output \( y = h(x) \), we compute the reduced Gröbner basis of \( \{ u - \sum_{|\alpha| = 1} a_{\alpha} x^{\alpha}, y - h(x) \} \subset \mathbb{R}[y,u,x] \) with a term ordering with \( y \prec x_1 \prec x_2 \prec u \). Note that a term ordering is arbitrary as long as an ordering of variables is \( y \prec x_1 \prec x_2 \prec u \). If the reduced Gröbner basis contains a polynomial \( u + p(y) \), the feedback controller can be obtained as an output feedback controller \( u = -p(y) \). In this example, the reduced Gröbner basis of

\[
\{ u + (x_1 + x_2) + (x_1 + x_2)^2, y - (x_1 + x_2) \}
\]

with the lexicographical ordering computed by using the function \texttt{plex} of Maple 18 is

\[
\{-y + x_1 + x_2, y^2 + u + y\},
\]

which gives an output feedback controller \( u = -y - y^2 \).

### 3.3 Reachability of invertible systems

Our results of dead-beat controllability can be utilized for reachability analysis of (globally) invertible systems. First, we show definitions of invertibility and reachability.

**Definition 3.3.** [6] The system \( \Sigma \) is called (globally) invertible if there exists \( p(z,u) := [p_1, \ldots, p_n] \in \mathbb{R}[z,u]^n \) such that

\[ p_i(f(x,u),u) = x_i \ (i = 1, \ldots, n) \]

and

\[ f_i(p(z,u),u) = z_i \ (i = 1, \ldots, n). \]

Moreover, the system

\[
z(t + 1) = p(z(t),v(t))
\]

is called an inverse-time system of \( \Sigma \), and \( p \) is denoted by \( f^{-1} \).

Invertibility can be tested by computing a suitable Gröbner basis owing to the following proposition.

**Proposition 3.1.** [6] A system is invertible if and only if one of the bases of ideal

\[ I := \langle z_1 - f_1(x,u), \ldots, z_n - f_n(x,u) \rangle \subset \mathbb{R}[z,u,x] \]

is of the form

\[ G = \{ x_1 - p_1(z,u), \ldots, x_n - p_n(z,u) \} \subset \mathbb{R}[z,u,x]. \]

We define reachability.
Definition 3.4. The system $\Sigma$ is said to be (globally) reachable from the origin if there exists a natural number $N$ such that for any terminal state $x_f \in \mathbb{R}^n$ there exists an input sequence $U_t = (u_1, \ldots, u_t) \in \mathbb{R}^t$ satisfying

$$f_{u_t, \ldots, u_1}(0) = x_f \quad (t \geq N).$$

Definition 3.5. [7] The system $\Sigma$ is said to be completely (globally) controllable if there exists a natural number $N$ such that for any initial state $x_0 \in \mathbb{R}^n$ and any terminal state $x_f \in \mathbb{R}^n$ there exists an input sequence $U_t = (u_1, \ldots, u_t) \in \mathbb{R}^t$ satisfying

$$f_{u_t, \ldots, u_1}(x_0) = x_f \quad (t \geq N).$$

Remark 3.1. We notice that the system is completely controllable if and only if its inverse-time system is dead-beat controllable and reachable from the origin.

Suppose that the system $\Sigma$ has an inverse-time system (7). If the inverse-time system is dead-beat controllable, for any initial state $z_0 \in \mathbb{R}^n$ there exists an input sequence $V_t = (v_1, \ldots, v_t) \in \mathbb{R}^t$ satisfying $p_{v_t, \ldots, v_1}(z_0) = 0 \ (t \geq N)$. From the definition of the inverse-time system, we have

$$f_{v_t}(0) = f(p_{v_t, \ldots, v_1}(z_0), v_t) = p_{v_{t-1}, \ldots, v_1}(z_0),$$

$$f_{v_{t-1}, v_t}(0) = f(p_{v_{t-1}, \ldots, v_1}(z_0), v_t) = p_{v_{t-2}, \ldots, v_1}(z_0),$$

$$\vdots$$

$$f_{v_1, \ldots, v_t}(0) = z_0.$$

Therefore, for the original system, there is an input sequence $U_t = (u_1, \ldots, u_t) := (v_1, \ldots, v_t)$ which transfers the origin to terminal state $z_0$. In summary, we have the following theorems in which Theorem 3.2 can be used to check dead-beat controllability.

Theorem 3.3. Suppose that the system $\Sigma$ is invertible. The system $\Sigma$ is reachable from the origin if and only if its inverse-time system is dead-beat controllable.

Theorem 3.4. Suppose that the system $\Sigma$ is invertible. The system $\Sigma$ is completely controllable if and only if both the system and its inverse-time system are dead-beat controllable, or equivalently, if and only if the inverse-time system is completely controllable.

Example 3.2. Consider

$$x_1(t+1) = x_1(t)(x_1^2(t) - x_3(t)) + x_2(t) - u_1(t) + x_1(t)u_2(t),$$

$$x_2(t+1) = x_1(t),$$

$$x_3(t+1) = -x_1^2(t) + x_3(t) - u_2(t).$$

First, we check invertibility. Define the ideal $I \subset \mathbb{R}[z, u, x]$:

$$I = \langle z_1 - x_1(x_1^2 - x_3) - x_2 + u_1 - x_1u_2, z_2 - x_1, z_3 + x_1^2 - x_3 + u_2 \rangle.$$

By computing the reduced Gröbner basis of $I$ with respect to the lexicographical ordering with $u_1 \prec u_2 \prec z_1 \prec z_2 \prec z_3 \prec x_1 \prec x_2 \prec x_3$, we have

$$I = \langle -z_2 + x_1, -z_2z_3 - u_1 + x_2 - z_1, -z_2^2 + x_3 - u_2 - z_3 \rangle.$$

Thus, from Proposition 3.1, the system is invertible. Note that invertibility can be verified by using another term ordering with $u_1 \prec u_2 \prec z_1 \prec z_2 \prec z_3 \prec x_1 \prec x_2 \prec x_3$. We check reachability from the origin of the original system, i.e. dead-beat controllability of the inverse-time system. The inverse-time system is computed as

$$z_1(t+1) = z_2(t),$$

$$z_2(t+1) = z_1(t) + z_2(t)z_3(t) + v_1(t),$$

$$z_3(t+1) = z_2^2(t) + z_3(t) + v_2(t),$$

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which is dead-beat controllable by a feedback controller

\[
v_1 := -z_1 - z_2 z_3, \\
v_2 := -z_2^2 - z_3.
\]

That is, the original system is reachable from the origin. The feedback controller is helpful to compute an input sequence of the original system, which transfers the origin to \((1,1,1)^T\). From the feedback controller, an input sequence of the inverse-time system, which transfers \((1,1,1)^T\) to the origin, is \(V_2 = ((-2,-2)^T,(-1,0)^T)\). Therefore, the input sequence for the original system is \(U_2 = ((-1,0)^T,(-2,-2)^T)\). Finally, we notice that the system is completely controllable because the original system is dead-beat controllable by a feedback controller

\[
u_1 := x_2, \\
u_2 := -x_1^2 + x_3.
\]

By using two feedback controllers, we can compute an input sequence which transfers any initial state to any terminal state.

**4. Conclusion**

In this paper, we derived a (global) dead-beat controllability condition based on algebraic geometry. This condition can be used for testing (global) reachability and complete controllability of (globally) invertible systems. Every property including invertibility can be verified by finite computations of polynomials.

**Acknowledgments**

This work was partly supported by JSPS KAKENHI Grant Numbers 15K18087 and 15H02257.

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