Special Article

A model of a nested small-world network *

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Abstract
The “small-world experiments” by Milgram et al. and the “β model” of small-world networks by Watts et al. are reviewed. Based on the criticism on the “small-world problem” provided by Kleinfeld, a model of a large-scale acquaintance network is constructed under the assumption that the stratified attributes of the nodes affect network formation. The model possesses a feature of self-similarity where connection of several local small-world networks forms a nested small-world network in global.

Keywords: social networks, small-world networks, fractal networks, spatial networks, self-similarity.

1. Introduction

“Small-world phenomenon” or more well known as “six degrees of separation” seems to become a common belief in the modern world. An idea of “small world”, all the people are connected by much shorter steps than we normally imagine, gives us some sense of security.

However, is the world really that small? Moreover, how small is it if it is small? Such question has been called “small-world problem” in the field of social sciences.

In this paper, firstly, the “small-world experiments” conducted by Milgram et al. (Milgram 1967, Travers and Milgram 1969, Korte and Milgram 1970) and the “β model” of small-world networks constructed by Watts et al. (Watts and Strogatz 1998, Watts 1999a, Watts 1999b, Watts 2003) are reviewed. Then, some problems that these small-world experiments bear are reconsidered through the criticisms by Kleinfeld (2002a, 2002b), and some problems of the β model are argued. Finally, construction of a model of a nested small-world network is shown. Discussions are given in the end.

2. Milgram’s small-world experiments

There is an experiment that uses a technique of “chain letters” called “small-world method”. Milgram is the one who invented and used this technique to tackle the “small-world problem”.

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In the experiment, those who are called “starters” are asked to start a relay of chain letters in order to eventually reach someone called “target” whom the starters do not know. The starters and intermediaries in the experiments pass a booklet to his/her acquaintances who are guessed to be closer to the target in geographical and/or social sense (for instance, occupation, etc.). The acquaintances in this experiment mean someone whom the starters and intermediaries know on a first name basis. Milgram let the starters and intermediaries know the name, occupation, and workplace of the target, as well as, the school from which the target graduated, when graduated, the date of military service, and target’s wife’s former name and birthplace.

Of course, not all the chain letters were completed through the target. However, surprisingly enough, Milgram repeatedly obtained the results in which the mean values of a number of intermediaries, necessary to reach the target in the completed chain letters, are about five. In a word, two people who do not know each other are connected in about “six degrees of separation”. It has been considered that these results showed the validity of a rule of thumb, “small-world phenomenon” or “six degrees of separation”.

There are three experiments related to the small-world problem done by Milgram et al. (Milgram 1967, Travers and Milgram 1969, Korte and Milgram 1970). However, due to the restriction of the space of the journal, let us consider the first experiment conducted by Milgram (1967) in the following.

In the first small-world experiment (Milgram 1967), Milgram chose two starters from two towns named Wichita in Kansas and Omaha in Nebraska. The former will be called “the Kansas study”, and the latter be called “the Nebraska study” as Milgram did in his paper. The target in the Kansas study is the wife of a divinity school student lived in Cambridge, Massachusetts. The target in the Nebraska study is a stockbroker worked in Boston, Massachusetts, and lived in Sharon, a suburb of Boston.

Milgram did not show any detailed data in his Kansas study. However, he reported, with his surprise, that there was a case in which only two intermediaries were needed between the starter and the target in a completed experiment. In short, that was three degrees of separation. In this case, the starter, who was a wheat farmer in Kansas, passed a booklet on an Episcopalian minister in his hometown. Then, this minister sent it to the minister who used to be his student in Cambridge, and he delivered it to the target who is the wife of a divinity school student.

Next, let us examine the Nebraska study. In the Nebraska study, 160 starters were sampled, that is, 160 chain letters existed. According to Milgram (1967), 44 chains (27.5%) had reached the target. Figure 1, which is a reproduction of a figure in Milgram’s paper (1967, p.65), shows distribution of the number of intermediaries in those 44 completed trials. Milgram reported the median of the number of intermediaries was five. The mean value can be calculated as 5.43. “Six Degrees of Separation” was born here.
3. A model of small-world network by Watts & Strogatz (β model)

Followed by Milgram's experiments, Watts and Strogatz (Watts and Strogatz 1998, Watts 1999a, Watts 1999b, Watts 2003) constructed a breakthrough model that is said to successfully explain the mechanism of small-world phenomenon. They named the model as “β model”. In their β model, they started with a ring-like lattice network (one-dimensional lattice network) with N nodes, which is highly clustered in local, and then conducted random rewiring on the edges of the network. What they showed was that a very small number of random rewiring (about 1 to 10% of the total edges) can dramatically decrease the characteristic path length, which measures the number of steps between two arbitrarily selected nodes in the network.

There are two steps in the procedure of random rewiring. The first step is to decide whether an edge be randomly rewired with a probability given as β, which is treated as a parameter in the “β model”. The second step is to choose where the selected edge to be rewired out of N-1 nodes with uniform probability 1/(N-1). They called this a “relational graph”.

As long as the number of randomly rewired edges is kept small, the number of local clusters in the network, which can be measured by the clustering coefficient, is preserved as high. While, only characteristic path length shows sharp decline. In other words, “small-world phenomena” are realized by a very small number of random connections that become random shortcuts. The definition of the small-world network by Watts is given as “highly clustered graphs with small characteristic path lengths” (Watts 2003, p42).

Here, note that clustering coefficient of a network, C, is defined as an averaged value of clustering coefficient of each node, n, in the network, say, c_n. The value of c_n can be calculated as dividing the number of total edges in a subgraph that consists of nodes adjacent to the node n excluding n itself by the total number of possible edges in the subgraph, which is obtained as k_n(k_n-1)/2 where k_n is the number of edges connecting to the node n. Characteristic path length, L, is defined as mean value of...
shortest steps between any two nodes in the network.

Figure 2 shows the clustering coefficient, C(β), and the characteristic path length, L(β), of the randomly rewired one-dimension lattice network that consist of 200 nodes (N=200), 400 edges (K=400) and average degree k=4. The probability, β, is treated as a tuning parameter. These values, C(β) and L(β), are averaged out in ten-time trials and depicted as <C(β)> and <L(β)> respectively. Note that, in the figure, these averaged values are standardized by being divided by C(0) and L(0), such as <C(β)>/C(0) and <L(β)>/L(0), respectively. The horizontal axis is logged, and the bars show ±σ (standard deviation). The dashed line shows the index of the small-world phenomenon, which equals to Log(200)/L(0)=0.2087 (Masuda 2006, p.36). In the figure, a small-world network is observed when β is around 0.1.

Figure 2: The averaged and standardized values of the clustering coefficient, <C(β)>/C(0), and characteristic path length, <L(β)>/L(0) are shown here. A small-world network is observed when β is around 0.1.

4. Problems of the small-world experiments

Kleinfeld (2002a, 2002b) has thrown doubt on validity of the small-world experiments done by Milgram et al. (Milgram 1967, Travers and Milgram 1969, Korte and Milgram 1970). She has reviewed these experiments by using the archives on Milgram (Kaplan 1996) in the library at the Yale University, and pointed out two main problems as follows.

According to Kleinfeld (2002a, p.63), the advertisement, which was used to recruit starters in the Kansas study, was worded so as to attract “not representative people but particularly sociable people proud of their social skills and confident of their powers to reach someone across class barriers”. In addition, she has discovered that, in the Nebraska study, Milgram used mailing lists in order to recruit subjects in the experiments. Those who were listed in such mailing lists were more likely to be rich people and therefore are more connected, that is, more likely to be able to make the steps in the
Another problem that is pointed out by Kleinfeld (2002a) is related to low frequency of completion of the chain letters. For instance, she found out that completion of the chain letters in the Kansas study was 5%, only 3 out of the 60 booklets reached the target.

Like the nerve network of “C. elegans”, the coactor network of the movie stars, and the electric network in North America, as Watts (Watts and Strogatz 1998, Watts 1999a, Watts 1999b, Watts 2003) described, there exist small-world networks in the world. Moreover, as Kleinfeld (2002a) mentioned, there exist small-world networks within a certain limited region such as acquaintance networks in a business field (Lundberg 1975), at a university (Shotland 1976), in a high-rise apartments (Bochner et al. 1976), and inside a certain city district (Lin et al. 1978). However, existence of these small-world networks, of course, does not ensure that a large-scale acquaintance network like Milgram was dealing with is also small-world network. These must be treated as separate questions.

On the other hand, after the criticism by Kleinfeld (2002a, 2002b), Watts et al. (Watts 2003, Dodds et al. 2003, Small World Project) conducted “by far the largest ever small-world experiment”. They recruited more than 60,000 email users over the world and asked them to do chain E-mails aiming to the 18 target people living in 13 distinct countries. Out of 24,163 relays that were actually started, 384 reached the targets. The completion rate is just 1.6%. From these data, they estimated the median of the number of degrees as 5 to 7. In a word, they seem to want to conclude that it is a small world still. However, as Watts himself said “almost certainly, the people who have access to a computer (and enough spare time to use it) represent a relatively narrow cross section of global society (Watts 2003, p135)”, these samples also have bias.

5. A fractal model of a small world

5.1 A fractal network and two ways of random rewiring

The world might not be as “small” as Milgram and Watts describe if what Kleinfeld insists is right to the point. Then, Watts’ β model, that is, the model based on one-dimensional lattice graph and “relational graph” might have to be renewed.

Firstly, one dimension lattice network in the β model is structurally homogeneous. Therefore, it disregards the influence of attributes of the nodes. Secondly, “relational graph” which is made by random rewiring under the uniform probability distribution disregards the influence of social distance, cultural distance, and geographic distance that may be related to the attributes. It is conjectured that these two bases in the β model play a role of “dramatic” decrease of average path length. This conjecture is examined by introducing the fractal β model discussed as follows.

First of all, in the fractal β model, as shown in Figure 3, the network that has fractal structure is introduced. The number of total nodes and total edges are given as $N=3^6=729$ and $K=3(3^6-1)/2=1092$, respectively. The value of average edges is calculated as $k=2.9958$. The average path length and
clustering coefficient are obtained as $L(0)=33.41582$ and $C(0)=0.33608$, respectively. Introducing the network with fractal structure represents the influence of stratified attributes. As in the figure, there are six levels of attributes.

Second, in the fractal $\beta$ model, as Watts et al. (Watts and Strogatz 1998, Watts 1999a, Watts 1999b, Watts 2003) did, random rewiring of the edges is carried out. The parameter $\beta$ shows the ratio of randomly rewired edges to the total edges. Here, importantly, there are two ways of random rewiring of the edges. One way is that random rewiring is done under the uniform probability distribution. This is the same way of random rewiring as in “relational graph”, i.e., the levels of attributes in the fractal network do not affect the probability in the procedure of random rewiring. In the other way, the levels of attributes influence the probability in the random rewiring process, and this is known as “spatial graph” (Watts 1999b). As an instance of the idea of the “spatial graph”, Wong (2006) showed that physical distance gives significant influences on the construction of a social network.

5.2 The fractal $\beta$ model with “relational network”

Random rewiring under the uniform probability distribution is conducted on the fractal network shown in the figure 3. This process makes “relational network”.

Figure 4 (a1) describes the picture of the whole network $(N=3^6)$ when $\beta$ is set as 0.1. In the figure 4 (a2), which is a redrew picture of figure 4 (a1) by the Kamada-Kawai optimization algorithm in Pajek, one can see that the nodes that bear distinct attributes are mixed up and it seems to contain a feature.
A model of a nested small-world network such as Watts’ $\beta$ model. This is simply because the probability in the process of random rewiring in this case is not affected by the levels of attributes. Figure 4 (b1) depicts the picture of the partial network ($N=3^5$) in the figure 4 (a1). Figure 4 (b2) is a redrew picture of figure 4 (b1) by the Kamada-Kawai optimization algorithm, likewise. Here, importantly, this partial network or local network dose not look like a small-world network.

Figure 5 (a) and (b) show the clustering coefficient $\langle C(\beta) \rangle / C(0)$ and characteristic path length $\langle L(\beta) \rangle / L(0)$ of the whole ($N=3^6$) and partial ($N=3^5$) network, respectively. Figure 5 (a) corresponds to figure 4 (a1) and (a2), and figure 5 (b) corresponds to figure 4 (b1) and (b2). The dashed lines in figure 5 (a) and (b) show the indices of the small-world phenomenon (Masuda 2006, p.36).

From the above figures, one can see that, under the process of “relational graph”, we have a network that is almost small world in global but not a small world in local. This may be an unrealistic model for an acquaintance network especially when we consider those cases such as Lundberg (1975), Shotland (1976), Bochner et al. (1976), and Lin et al. (1978), where there exist small-world networks in local.

Figure 4: The fractal $\beta$ model with “relational network”. The picture of the whole network in (a1) and (a2), and the partial network in (b1) and (b2).
Figure 5: The clustering coefficient $\langle C(\beta) \rangle / C(0)$ and characteristic path length $\langle L(\beta) \rangle / L(0)$ of the whole (a) and partial (b) networks that correspond to figure 4 (a’s) and (b’s), respectively, are shown here.

5.3 The fractal $\beta$ model with “spatial network”

In the following model, the levels of attributes in the fractal network affect the probability in the process of random rewiring. More precisely, for example, those nodes that have level-2 attribute are more likely to be connected with the nodes with the same level of the attribute in the process of random rewiring. In the following case, when the edges are randomly rewired, the probability of the nodes in the same level being selected to be connected is set as 0.9.

Figure 6 (a1) describes the picture of the whole network ($N=3^5$) when $\beta$ is set as 0.1. In the figure 6 (a2), a redrew picture of figure 6 (a1), one can see that the nodes that have distinct attribute are ‘not’ mixed up because the above-mentioned probability related to the levels of attributes. Figure 6 (b1) depicts the picture of the partial network ($N=3^5$) in the figure 6 (a1) and figure 6 (b2) is a redrew picture of figure 4 (b1). Here, this partial network or local network looks like a small-world network.

Figure 7 (a) and (b) show the clustering coefficient $\langle C(\beta) \rangle / C(0)$ and characteristic path length $\langle L(\beta) \rangle / L(0)$ of the whole ($N=3^5$) and partial ($N=3^5$) network, respectively. Figure 7 (a) corresponds to figure 6 (a1) and (a2) and figure 7 (b) corresponds to figure 6 (b1) and (b2). The dashed lines in figure 7 show the index of the small-world phenomenon (Masuda 2006, p.36).

From the above figures, importantly, in the process of “spatial graph”, we have a network that is ‘almost’ small world in both global and local. In this case, the whole (global) network has become small world because it consists of several partial (local) small-world networks. It can be said that this is a model of nested small-world network and that this may be a realization of the model that is suggested by Kleinfeld as in “[t]he ‘lumpy oatmeal’ theory, that we live in a world with many small worlds possibly but not necessarily connected, might be viewed as the “weak” form of the small world phenomenon, for which we do have evidence.” (Kleinfeld 2002a, p.65). The evidence in her suggestion corresponds to the small-world networks within a certain limited region such as acquaintance networks in a business field (Lundberg 1975) and so on, that are shown in the section 4...
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...in this paper.

(\text{a1}) \quad (\text{a2})

(b1) \quad (b2)

Figure 6: The fractal $\beta$ model with “spatial network”. The picture of the whole network in (a1) and (a2), and the partial network in (b1) and (b2).

(a) \quad (b)

Figure 7: The clustering coefficient $\langle C(\beta) \rangle / C(0)$ and characteristic path length $\langle L(\beta) \rangle / L(0)$ of the whole (a) and partial (b) network that correspond to figure 6 (a’s) and (b’s), respectively, are shown.
6. Discussion

In this paper, the “small-world experiments” (Milgram 1967, Travers and Milgram 1969, Korte and Milgram 1970) and the “β model” of small-world networks (Watts and Strogatz 1998, Watts 1999a, Watts 1999b, Watts 2003) are reviewed. The criticisms toward the experiments raised by Kleinfeld (2002a, 2002b) are also reviewed, and some problems of the β model are argued. Finally, a model of a nested small-world network is constructed.

In the model of a nested small-world network, because of the process of “spatial graph”, we have a network that is almost small world in both global and local. In this model, the whole (global) network has become small world because it consists of several partial (local) small-world networks. On the other hand, under the process of “relational graph”, we have an unrealistic model of a social network, that is, this process produces a network which is almost small world in global but not a small world in local (see Table 1).

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<th>Local (Partial)</th>
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<td>Spatial Graph</td>
<td>Small World</td>
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<td>Relational Graph</td>
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Table 1

It is important to understand the structure of an acquaintance network correctly because, for instance, it helps us to understand the dynamics of information diffusion in the network. Therefore, we care if an acquaintance network is small world or not. Regarding to this point, the author is currently involved with an ongoing project where an original social network service, named TOMOCOM, is built and data related to the social networks of students from several universities in Japan are being collected. Testing whether there exists a feature of self-similarity in the networks as in a model of a nested small-world network is now underway.

In addition to the above perspective, there is a different viewpoint when one cares if it is small world or not. As Kleinfeld pointed out, an idea of “small world” gives us some sense of security because this idea tends to be understood as that all the people on the planet are connected by much shorter steps than we normally imagine. However, it is also important to think about the influences which the level of attributes, income gap, prejudice, social distance, cultural distance, and so on may give effects on social networks. These influences may make the world big and even disconnected.
References


TOMOCOM. http://tomocom.jp/


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理論と方法