PAPERS COMMUNICATED

108. On the Wiener’s Formula.

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1. Wiener\(^1\) has proved that:

\[
\mathfrak{M}\{f\} = \lim_{x \to \infty} \frac{1}{x} \int_0^x f(\xi) d\xi
\]

exists and is finite and \(1/x \int_0^x \left| f(\xi) \right| d\xi \) is bounded in \((0, \infty)\), then

\[
\lim_{\varepsilon \to 0} \frac{1}{\pi} \int_0^x f(x) \frac{\sin \varepsilon x}{\varepsilon^2} dx = \mathfrak{M}\{f\}. \tag{1}
\]

Bochner\(^2\) has replaced the kernel \(\frac{\sin^2 x}{x^2}\) in (1) by a general function \(K(x)\) and found the conditions for the validity of

\[
\lim_{\varepsilon \to 0} \int_0^\infty f\left(\frac{x}{\varepsilon}\right) K(x) dx = \mathfrak{M}\{f\} \int_0^\infty K(x) dx. \tag{2}
\]

Bochner named (2) the Wiener’s formula.

In this paper, we treat the conditions of validity of (2).

2. Theorem 1. Suppose that (i) \(K(x)\) is absolutely continuous in any finite interval, (ii) \(K(x)\) is absolutely integrable in \((0, \infty)\), (iii) \(x K(x)\) is of bounded variation in \((0, \infty)\), and (iv) \(\frac{1}{x} \int_0^x f(\xi) d\xi\) is bounded in \((0, \infty)\), and (v) the limit \(\mathfrak{M}\{f\} = \lim_{x \to \infty} \frac{1}{x} \int_0^x f(\xi) d\xi\) exists and is finite. Then we have

\[
\lim_{\varepsilon \to 0} \int_0^\infty f\left(\frac{x}{\varepsilon}\right) K(x) dx = \mathfrak{M}\{f\} \int_0^\infty K(x) dx. \tag{2}
\]

Proof. Without loss of generality, we can suppose that

\[\]


2) Bochner, Vorlesungen über Fouriersche Integrale, 1933, pp. 30-32.
Then it is sufficient to prove
\[ \lim_{\varepsilon \to 0} \int_{0}^{\infty} f\left( \frac{x}{\varepsilon} \right) K(x) dx = 0 \]

instead of (2). And we can suppose that \( \varepsilon < 1 \).

If we put \( F(x) = \int_{0}^{x} \frac{f(\xi)}{\xi} d\xi \), we have, by integration by parts and the condition (i),
\[
\int_{A}^{B} f\left( \frac{x}{\varepsilon} \right) K(x) dx = \left[ K(x) \int_{x}^{A} f\left( \frac{\xi}{\varepsilon} \right) d\xi \right]_{x=A}^{x=B} - \int_{A}^{B} K'(x) F\left( \frac{x}{\varepsilon} \right) dx
\]
\[
= K(B) F\left( \frac{B}{\varepsilon} \right) - K(A) F\left( \frac{A}{\varepsilon} \right) - \varepsilon \int_{A}^{B} K'(x) F\left( \frac{x}{\varepsilon} \right) dx
\]
\[
= BK(B) F\left( \frac{B}{\varepsilon} \right) - AK(A) F\left( \frac{A}{\varepsilon} \right) - \varepsilon \int_{A}^{B} K'(x) F\left( \frac{x}{\varepsilon} \right) dx
\]

By the condition (iii), \( xK(x) \) is bounded in \( (0, \infty) \), that is, there is a constant \( M \) such that \( |xK(x)| < M \). By (3), for any \( \gamma > 0 \), we can find \( A_0 \) such that \( \frac{F(x)}{x} < \gamma \) for \( x > A_0 \). Then we have
\[
\left| \int_{A}^{B} f\left( \frac{x}{\varepsilon} \right) K(x) dx \right| \leq 2\gamma M + \gamma \int_{A}^{B} |xK'(x)| dx \tag{4}
\]
for \( A > A_0 \). By the identity
\[
\frac{d}{dx} \{ xK(x) \} = K(x) + xK'(x)
\]
and the conditions (i), (ii) and (iii), \( xK'(x) \) is absolutely integrable in \( (0, \infty) \), then there is an \( N \) such that \( \int_{0}^{\infty} |xK'(x)| dx < N \). Therefore, letting \( B \to \infty \) in (4), we have
\[
\left| \int_{A}^{\infty} f\left( \frac{x}{\varepsilon} \right) K(x) dx \right| \leq (2M + N) \gamma \tag{5}
\]
for an \( A > A_0 \).

We have, by (i) and integration by parts,
\[
\int_{0}^{A} f\left( \frac{x}{\varepsilon} \right) K(x) dx = K(A) F\left( \frac{A}{\varepsilon} \right) - \varepsilon \int_{0}^{A} K'(x) F\left( \frac{x}{\varepsilon} \right) dx \tag{6}
\]
If we take a such that \( \int_0^a |xK'(x)| \, dx < \gamma \), and \( L \) such that \( \frac{|F(x)|}{x} < L \) for \( x \) in \((0, \infty)\), then

\[
|\varepsilon \int_0^a K'(x) F\left( \frac{x}{\varepsilon} \right) \, dx | \leq L |xK'(x)| \, dx < \gamma L ,
\]

and there is an \( \varepsilon_0 \) such that \( \frac{\varepsilon}{x} F\left( \frac{x}{\varepsilon} \right) \) for any \( x > a \) and any \( \varepsilon < \varepsilon_0 \). Hence

\[
|\varepsilon \int_0^A K'(x) F\left( \frac{x}{\varepsilon} \right) \, dx | \leq \int_0^A |xK'(x)| \cdot \frac{\varepsilon}{x} F\left( \frac{x}{\varepsilon} \right) \, dx
\]

\[
\leq \gamma \int_0^A |xK'(x)| \, dx
\]

\[
\leq \gamma \int_0^\infty |xK'(x)| \, dx
\]

\[
\leq \gamma N .
\]

We have, by (6), (7) and (8),

\[
\int_0^A f\left( \frac{x}{\varepsilon} \right) K(x) \, dx \leq (M + L + N) \gamma
\]

for \( \varepsilon \leq \varepsilon_0 \). Thus (2') is proved.

3. Similarly we can prove the following theorem.

**Theorem 2.** Suppose that (i) \( K(x) \) is absolutely continuous in any finite interval and there is a function \( K(x) \) such that (ii) \( |K(x)| \leq K(x) \), (iii) \( K(x) \) is absolutely continuous in any finite interval and absolutely integrable in \((0, \infty)\), (iv) \( xK(x) \) is of bounded variation in \((0, \infty)\) and tends to zero as \( x \to \infty \). Further suppose that (v) \( \frac{1}{x} \int_0^x |f(\xi)| \, d\xi \) is bounded in \((0, \infty)\), and (vi) the limit

\[
\mathfrak{M}\{f\} = \lim_{x \to \infty} \frac{1}{x} \int_0^x f(\xi) \, d\xi
\]

exists and is finite. Then we have

\[
\lim_{\varepsilon \to 0} \int_0^\infty f\left( \frac{x}{\varepsilon} \right) K(x) \, dx = \mathfrak{M}\{f\} \int_0^\infty K(x) \, dx .
\]

This is a generalization of the Bochner's theorem.

4. Further we can prove the following theorem.

**Theorem 3.** Let \( k \) be a positive integer. Suppose that (i) \( K(x) \) is absolutely integrable in \((0, \infty)\), (ii) \( K^{(k-1)}(x) \) is absolutely continuous in any finite interval, (iii) \( x^i K^{(i-1)}(x) \) is of bounded varia-
tion in $(0, \infty)$, and (iv) $x^i K^{(i-1)}(x)$ $(i=1, 2, \ldots, k-1)$ tends to zero as $x \to \infty$. Further suppose that (v) $\frac{1}{x} \int_0^x f(\xi) d\xi$ is bounded in $(0, \infty)$, and (vi)

$$\mathcal{M}_k \{f\} = \lim_{x_1 \to \infty} \frac{1}{x_1} \int_0^{x_1} \int_0^{x_2} \int_0^{x_3} \cdots \int_0^{x_k} f(\xi) d\xi$$

exists and is finite. Then we have

$$\lim_{\varepsilon \to 0} \int_0^\infty f\left(\frac{x}{\varepsilon}\right) K(x) dx = \mathcal{M}_k \{f\} \int_0^\infty K(x) dx.$$