101. A Theorem Concerning the Fourier Series of a Quadratically Summable Function.

By Tatsuo KAWATA.

Mathematical Institute, Tohoku Imperial University, Sendai.

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1. Recently Mr. R. Salem\(^1\) has proved the following theorem:

If \( f(x) \) is a bounded periodic function with period \( 2\pi \) and its Fourier coefficients are \( a_n, b_n \), then the following relation holds for almost all values of \( x \),

\[
\lim_{s \to 0} \left[ \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{1 + s\sqrt{\log n}} \right] = f(x).
\]

Actually he proved the relation (1) replacing more general sequence \( \{\psi_n(s)\} \) for \( \{1/(1+s\sqrt{\log n})\} \). The object of the present paper is to prove the validity of (1) under the condition that \( f(x) \in L_2 \), i.e. is quadratically summable. In this form the theorem says more than the well known theorem of Kolmogoroff-Seliverstoff-Plessner\(^2\) concerning the convergence factor of the Fourier series of a quadratically summable function. But we can prove our theorem by using the theorem of Kolmogoroff-Seliverstoff-Plessner.

2. Theorem 1. If \( f(x) \in L_2 \) and is periodic with period \( 2\pi \) and \( a_n, b_n \) are its Fourier coefficients, then the relation (1) holds for almost all values of \( x \).

Theorem 2. In Theorem 1, we can replace the sequence \( \{1/(1+s\sqrt{\log n})\} \) by the sequence \( \{\psi_n(s)\} \) which satisfies the following conditions:

1°. \( \{\psi_n(s)\} \) is the decreasing and convex sequence of positive functions, \( 0 < s < 1 \) (\( \psi_0(s) = 1 \)).

2°. \( \lim_{s \to 0} \psi_n(s) = 1 \), \( (n \text{ fixed}) \).

3°. \( \lim_{n \to \infty} \psi_n(s) = 0 \), \( (s \text{ fixed, } > 0) \).

4°. \( \psi_n(s) = O \left( \sqrt{\log n} \right) \), \( (s \text{ fixed, } > 0) \).

5°. \( \psi_n(s) \) has a finite number of maxima for any fixed \( n \).

The proof of Theorem 2 is quite similar as that of Theorem 1 and so we only prove Theorem 1.

Let \( E_1 \) be the set of \( x \) such that

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1) R. Salem, Sur une méthode de sommation, valable presque partout, pour les séries de Fourier de fonction continue, Comptes Rendus, 205 (1937), pp. 14-16.

converges. Then $mE_1 = 2\pi$. This is the theorem of Kolmogoroff-Seliverstoff-Plessner. And for $x \in E_1$,
\[
\sum_{n=2}^{N} (a_n \cos nx + b_n \sin nx) = o(\sqrt[\log N]) .
\]
We can easily verify that for $x \in E_1$,
\[
f(x) = \frac{1}{2} a_0 + \frac{\sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx}{1 + s\sqrt{\log n}}
\]
converges and its $N$-th partial sum is $o(\sqrt[\log n])$ for every value of $s$.

The Parseval relation shows that
\[
\lim_{\pi \to 0} \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x, s) - f(x)|^2 dx = \lim_{s \to 0} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \left( \frac{1}{1 + s\sqrt{\log n}} - 1 \right)^2 = 0 .
\]

From the known result concerning the convergence in mean, we see that there exists a sequence $\{s_n\}$ (lim $s_n = 0$) such that
\[
\lim_{s \to \infty} f(x, s_n) = f(x)
\]
for almost all values of $x$.

Now let $f(x) = f^+(x) - f^-(x)$, where
\[
f^+(x) = f(x), \quad f^-(x) = -f(x) ,
\]
for $x \in S_1$, and $= 0$, otherwise.

Then $f^+(x) \geq 0$ and $f^+(x), f^-(x) \leq |f(x)|$.

Write
\[
f^+(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + \beta_n \sin nx) ,
\]
\[
f^-(x) = \frac{1}{2} \gamma_0 + \sum_{n=1}^{\infty} (\gamma_n \cos nx + \delta_n \sin nx) ,
\]
then clearly $a_n - \gamma_n = a_n, \beta_n - \delta_n = b_n$. Similar arguments as above show that there exist a set $S_1$ and a sequence $\{s_n\}$ such that $mS_1 = 2\pi$ and for $x \in S_1, f^+(x, s)$ converges and the $N$-th partial sums are $o(\sqrt[\log N])$ and $\lim_{n \to \infty} f^+(x, s_n) = f^+(x)$. By applying the Abel's transformation twice, we have, if $x \in S_1$

\[
(3) \quad f^+(x, s) = \lim_{N \to \infty} \left\{ \frac{1}{2} a_0 + \sum_{n=1}^{N} \frac{a_n \cos nx + \beta_n \sin nx}{1 + s\sqrt{\log n}} \right\}
\]

\[
(4) \quad = \lim_{N \to \infty} \left\{ \frac{1}{2} \sum_{n=0}^{N-2} K_n(x) J^2 \frac{1}{1 + s\sqrt{\log n}} + K_{N-1}(x) J^2 \frac{1}{1 + s\sqrt{\log (N-1)}} + S_N(x) \frac{1}{1 + s\sqrt{\log N}} \right\} ,
\]
where $S_n(x)$ is the $N$-th partial sum of the series in the bracket of the right hand side of (3) and

$$K_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+t) \frac{\sin^2(nt/2)}{\sin^2(t/2)} dt, \quad (n > 0), \quad K_0(x) = a_0$$

and

$$\Delta a_p = a_p - a_{p+1}.$$ 

In this, we must replace 1 for $\frac{1}{1 + sv \log n}$ if $n = 0$. The last term in the bracket of (4) tends to zero as $N \to \infty$ and the same is also easily verified for the second term. Thus

$$f^+(x, s) = \sum_{n=0}^{\infty} K_n(x) \frac{1}{1 + s \sqrt{\log n}},$$

where we notice that $K_n(x)$ and $\Delta^2 \frac{1}{1 + s \sqrt{\log n}}$ are positive and $f^+(x, s)$ is also positive. Now take two numbers $s_p, s_{p+1}$ from $\{s_n\}$ such that $s_{p+1} \leq s \leq s_p$. Then we have

$$0 \leq f^+(x, s) \leq \sum_{n=0}^{\infty} K_n(x) \frac{1}{1 + s_p \sqrt{\log n}} + \sum_{n=0}^{\infty} K_n(x) \frac{1}{1 + s_{p+1} \sqrt{\log n}} + \sum_{n=2}^{n_{s+2}} K_n(x) \frac{1}{1 + s \sqrt{\log n}}$$

$$= f^+(x, s_p) + f^+(x, s_{p+1}) + \frac{1}{n_s} \int_0^{2\pi} |f(x+t)| \frac{\sin^2(n_s t/2)}{\sin^2(t/2)} dt,$$

for some $n_s$ which tends to $\infty$ as $s \to 0$.

Hence we have

$$\lim_{s \to 0} f^+(x, s) \leq 2f^+(x) \leq 2|f(x)|.$$

Similarly there exists a set $S_2$ such that for $x \in S_2$,

$$\lim_{s \to 0} f^-(x, s) \leq 3|f(x)|.$$

Thus for $x \in S_1 \cdot S_2$ we have

$$\lim_{s \to 0} |f(x, s)| \leq \lim_{s \to 0} f^+(x, s) + \lim_{s \to 0} f^-(x, s) \leq 6|f(x)|.$$

Now let

$$f_M = \sum_{n=-M+1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

and

$$f_M(x, s) = \sum_{n=-M+1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{1 + s \sqrt{\log n}}, \quad (x \in E_1).$$

Then there exists a set $E_M$ such that $mE_M = 2\pi$ and for $x \in E_M$, 

Thus in \( E_M \), \( \lim_{s \to 0} |f_M(x, s)| \) is finite for every \( M \). Squaring and integrating both sides of (5), we have
\[
\int_{-\pi}^{\pi} \left( \lim_{s \to 0} |f_M(x, s)| \right)^2 dx \leq 6\pi \int_{-\pi}^{\pi} |f_M(x)|^2 dx = 6\pi \sum_{n=-M+1}^{\infty} (a_n^2 + b_n^2).
\]
Hence we get
\[
\lim_{M \to \infty} \int_{-\pi}^{\pi} \left( \lim_{s \to 0} |f_M(x, s)| \right)^2 dx = 0.
\]
Therefore there exist a set \( E \) and a sequence \( M_k \) such that \( mE = 2\pi \), and for \( x \in E \)
\[
\lim_{K \to \infty} \lim_{s \to 0} |f_{M_k}(x, s)| = 0.
\]
Now for \( x \in E \),
\[
\lim_{s, s' \to 0} |f(x, s) - f(x, s')| \leq \lim_{s, s' \to 0} \sum_{n=1}^{M_k} (a_n \cos nx + b_n \sin nx)
\times \left( \frac{1}{1 + s \sqrt{\log n}} - \frac{1}{1 + s' \sqrt{\log n}} \right)
+ 2 \lim_{s \to 0} \sum_{n=-M_k+1}^{\infty} (a_n \cos nx + b_n \sin nx) \frac{1}{1 + s \sqrt{\log n}}
= 2 \lim_{s \to 0} \sum_{n=-M_k+1}^{\infty} (a_n \cos nx + b_n \sin nx) \frac{1}{1 + s \sqrt{\log n}}
\]
which is arbitrarily small by taking \( k \) large. Thus \( \lim_{s \to 0} f(x, s) \) exists for \( x \in E \). The fact that the limiting value is \( f(x) \) is an immediate consequence of (2). Thus we complete the proof.