45. Gravity Anomalies and the Corresponding Subterranean Mass Distributions.

By Chuji TSUBOI.

Earthquake Research Institute, Tokyo Imperial University.

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1. From the purely mathematical point of view, the determination of subterranean mass distributions from the corresponding gravity values is an indeterminate problem, but it is often reduced to a determinate one, by means of assumptions which we may plausibly make regarding the mode of the mass distributions. If two materials of different densities are arranged in layers in the earth's crust with a boundary of small undulation, this mass distribution can approximately be replaced by that concentrated on the subterranean plane situated at the average depth of the boundary, so far as the gravity anomalies are concerned.

2. Let A in Fig. 1 be the earth's surface, and the gravity anomalies $\Delta g(xy)$ be measured on it from point to point, and B the plane at depth d over which the mass $\rho(xy)$ is distributed, responsible for $\Delta g(xy)$. Taking x - and y -axes on the surface A and z -axis vertically downward as shown in Fig. 1, the gravity potential U due to $\rho(xy)$ is the solution of Laplace equation

Fig. 1.

$$\nabla^2 U = 0, \dots\dots\dots (1)$$

with the conditions that

$$\text{at } z = 0, \quad -\frac{\partial U}{\partial z} = g(xy), \dots\dots\dots (2)$$

$$\text{and at } z = d, \quad -\frac{\partial U}{\partial z} = 2\pi k^2 \rho(xy), \dots\dots\dots (3)$$

where k^2 is the gravitation constant.

U assumes the form

$$U = -\sum \sum A_{mn} \frac{\cos mx \cos ny}{\sin mx \sin ny} e^{-\sqrt{m^2+n^2}(d-z)} \dots\dots\dots (4)$$

in the space above the plane B .

$$\text{If } \rho(xy) = \sum \sum B_{mn} \frac{\cos mx \cos ny}{\sin mx \sin ny} \dots\dots\dots (5)$$

$$\text{and } \Delta g(xy) = \sum \sum C_{mn} \frac{\cos mx \cos ny}{\sin mx \sin ny}, \dots\dots\dots (6)$$

then, from the boundary conditions, it follows that

$$\sum \sum \sqrt{m^2+n^2} A_{mn} \frac{\cos mx \cos ny}{\sin mx \sin ny} = 2\pi k^2 \sum \sum B_{mn} \frac{\cos mx \cos ny}{\sin mx \sin ny} \dots\dots (7)$$

and

$$\sum \sum \sqrt{m^2+n^2} A_{mn} \frac{\cos mx \cos ny}{\sin mx \sin ny} e^{-\sqrt{m^2+n^2}d} = \sum \sum C_{mn} \frac{\cos mx \cos ny}{\sin mx \sin ny} \dots (8)$$

Eliminating A_{mn} from (7) and (8), we get

$$B_{mn} = \frac{C_{mn}}{2\pi k^2} e^{\sqrt{m^2+n^2}d}, \dots (9)$$

which is the expression by means of which $\rho(xy)$ can be directly determined from $\Delta g(xy)$, provided d is known from other data or is assumed reasonably. In other words, the procedure of the determination is to develop $\Delta g(xy)$ into a double Fourier series, then to multiply each of the coefficients by the corresponding $\frac{1}{2\pi k^2} e^{\sqrt{m^2+n^2}d}$, and lastly, to sum up the new series.

The deviation of the vertical θ and the undulation of the geoid h can be obtained from $\Delta g(xy)$ by means of similar calculations. Since the gravitational forces acting in x - and y -directions along the earth's surface are respectively

$$X = -\left(\frac{\partial U}{\partial x}\right)_{z=0}, \quad Y = -\left(\frac{\partial U}{\partial y}\right)_{z=0}, \dots (10), (11)$$

the corresponding deviations of the vertical are given by

$$\theta_x = \frac{X}{g} = \mp \frac{1}{g} \sum \sum \frac{m}{\sqrt{m^2+n^2}} C_m \frac{\sin mx \cos ny}{\cos mx \sin ny}, \dots (12)$$

$$\theta_y = \frac{Y}{g} = \mp \frac{1}{g} \sum \sum \frac{n}{\sqrt{m^2+n^2}} C_{mn} \frac{\cos mx \sin ny}{\sin mx \cos ny} \dots (13)$$

The height of the geoid $h(xy)$ at x, y with respect to that at $x=x_0, y=y_0$ is obtained by integrating the deviation of the vertical along a suitable path; thus

$$h(xy) = \frac{1}{g} \sum \sum \frac{C_{mn}}{\sqrt{m^2+n^2}} \left\{ \frac{\cos mx \cos ny}{\sin mx \sin ny} - \frac{\cos mx_0 \cos ny_0}{\sin mx_0 \sin ny_0} \right\} \dots (14)$$

Were the distributions of gravity anomalies "two-dimensional," matters would be much simplified. Since in these cases, we may put $n=0$, we get

$$B_m = \frac{C_m}{2\pi k^2} e^{md}, \dots (15)$$

$$\theta_x(x) = \mp \frac{1}{g} \sum C_m \frac{\sin mx}{\cos mx}, \dots (16)$$

$$h(x) = \frac{1}{g} \sum \frac{C_m}{m} \left(\frac{\cos mx}{\sin mx} - \frac{\cos mx_0}{\sin mx_0} \right) \dots (17)$$

If a distribution of gravity gradients along the earth's surface be given, instead of that of gravity anomalies, there is a similar method

for finding the subterranean mass distribution, responsible for these gradient values. Thus, if

$$\frac{\partial g}{\partial x} = \sum \gamma_m \frac{\cos mx}{\sin mx}, \dots\dots\dots (18)$$

then

$$\rho(x) = \pm \frac{1}{2\pi k^2} \sum \frac{\gamma_m}{m} \frac{\sin mx}{\cos mx} e^{md} + \delta. \dots\dots\dots (19)$$

3. For the major problems in geophysics, in which the spherical co-ordinate should be used, the relation between the coefficients of surface spherical harmonic expansions of the gravity anomalies and those of the subterranean mass distributions can also be derived by the method which is similar to the foregoing one. Thus, if

$$\rho(\mu\varphi) = \sum B_m Y_m(\mu\varphi) \dots\dots\dots (20)$$

and

$$\Delta g(\mu\varphi) = \sum C_m Y_m(\mu\varphi), \dots\dots\dots (21)$$

where μ is the cosine of the co-latitude, φ , the azimuth, and Y_m , the surface spherical harmonics of m -th order, we get

$$B_m = \frac{1}{4\pi k^2} \frac{(2m+1) a^{m+2}}{(m+1)(a-d)^{m+2}} C_m \dots\dots\dots (22)$$

where a is the radius of the earth.

4. We shall secondly assume that, instead of being concentrated on a single plane, the subterranean mass is distributed in the vertical direction uniformly from the earth's surface down to depth D , so that the mass distribution along an elementary sheet at z is given by

$$d\rho(xy) = \frac{1}{D} \sum \sum B_{mn} \frac{\cos mx}{\sin mx} \frac{\cos ny}{\sin ny} dz. \dots\dots\dots (23)$$

The surface gravity anomalies due to the whole mass is then given by

$$\Delta g(xy) = \frac{2\pi k^2}{D} \sum \sum B_{mn} \frac{\cos mx}{\sin mx} \frac{\cos ny}{\sin ny} (1 - e^{-\sqrt{m^2+n^2}D}), \dots (24)$$

whence
$$B_{mn} = \frac{C_{mn}}{2\pi k^2} \sqrt{m^2+n^2} D / (1 - e^{-\sqrt{m^2+n^2}D}). \dots\dots\dots (25)$$

In the spherical co-ordinate, the corresponding relation is

$$B_m = \frac{D}{4\pi k^2} \frac{(2m+1)(m+3)}{(m+1)} \frac{a^{m+2}}{\{a^{m+3} - (a-D)^{m+3}\}} C_m. \dots\dots (26)$$

5. Some of the numerous applications of the foregoing method of calculation to actual problems are given in the following examples.

Example 1. The first application is concerned with the famous example which R. v. Eötvös¹⁾ showed for demonstrating the relation

1) R. v. Eötvös, Verh. 16 Allg. Conf. d. Intern. Erdmessung. (1910) 317.

between the gravity gradients and the subterranean mass distribution. Eötvös calculated the gravity gradients resulting from the mass distribution which he assumed to be as shown in Fig. 2. Following the method developed in the foregoing section, the gravity gradients were subjected to Fourier analysis and making use of the coefficients thus found, the subterranean mass distribution was determined, as if it had been unknown. The depth of the plane at which the mass is concentrated was assumed to be 500 m. The calculated values which are shown in Fig. 2 by filled circles agree satisfactorily with those assumed by Eötvös.

Fig. 2.

6. *Example 2.* The second example concerns with the gravity measurements in the Dutch East Indies made by Vening Meinesz.¹⁾

Fig. 3.

- (a) Isostatic Anomalies
- (b) Subterranean Mass
- (c) Geoid Undulation

The regional isostatic anomalies along his profile No. 21 were subjected to Fourier analysis and from the coefficients thus found, those for the mass distribution and the geoid undulation were calculated with the results shown in Fig. 3. In this case, d is assumed with Vening Meinesz to be 25 km. At the negative maximum, the mass ρ amounts to 5×10^5 . This is equivalent to a mass of density 0.6 and thickness 8 km., which result agrees exactly with what Vening Meinesz found from an indirect estimation.

7. *Example 3.* The surface topographical relief and the Bouguer anomalies along the profile No. 21 of Vening Meinesz were subjected to Fourier analysis with the results given in Table I. For the submarine topography at depth h , the height was taken to be $-0.62 h$, instead of $-h$, taking into consideration the mass of the sea water. In 21 out of 24 pairs of H_m and C_m , which are respectively Fourier coefficients for h and Δg , they differ in algebraic sign, indicating that the surface topography and the boundary between the crustal and the

1) Vening Meinesz, Gravity Expeditions at Sea, II (1934).

denser subcrustal materials are approximately opposite in relief as schematically shown in Fig. 4. If we take $\rho=2.7$, $-2\pi k^2 \rho H_0$ becomes 209×10^{-3} and this does not differ greatly from C_0 which is 155×10^{-3} . This state of things agrees with what Airy isostasy implies, although the isostatic anomalies are here not very small.

Fig. 4.

TABLE I. Fourier Coefficients for H_m and C_m across the Profile No. 21 in the Dutch East Indies.

m	H_m in m.		C_m in milligal	
	cos	sin	cos	sin
0	-1848		+155	
1	+ 617	-916	- 25	+118
2	- 102	-363	+ 24	+ 5
3	- 37	-223	- 14	+ 5
4	+ 13	+ 38	- 12	+100
5	+ 219	-131	- 8	+ 9
6	+ 44	-151	- 8	+ 6
7	+ 80	+ 14	- 7	+ 3
8	+ 88	- 91	- 6	+ 2
9	+ 56	- 5	- 7	+ 2
10	+ 87	- 18	- 7	+ 1
11	+ 61	- 6	- 8	
12	+ 43		- 3	

The similar study was made for the profile of the African Rift Valley, across which E. C. Bullard¹⁾ made gravity measurements. H_m and C_m found are given in Table II. In this case also, the main topography corresponding to the 0-th order is nearly compensated, since if we take $\rho=2.7$, we get

$$-2\pi k^2 \rho H_0 = -135 \times 10^{-3},$$

$$C_0 = -141 \times 10^{-3}$$

which are almost equal. In only 2 pairs of H_m and C_m , out of the remaining 11, they differ in algebraic sign, indicating that the surface topography and the boundary between the crustal and subcrustal materials are approximately the same in relief as schematically shown in Fig. 5. This shows that the structure of the rift continues down into a considerable depth.

Fig. 5.

1) E. C. Bullard, Phil. Trans. Roy. Soc. London, **235** (1936), 446.

TABLE II. Fourier Coefficients for H_m and C_m across the African Rift Valley.

m	H_m in m .		C_m in milligal	
	cos	sin	cos	sin
0	+1192		-141	
1	- 301	-514	- 3	-19
2	+ 329	+354	- 10	+22
3	+ 208	+142	+ 7	+58
4	+ 171	- 22	+ 6	- 1
5	+ 118	+ 1	+ 1	- 5
6	- 21		- 1	