55. Ultrashort Electromagnetic Waves Generated in a Vacuum Tube by Ordinary Electric Discharge.

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It was announced by Prof. S. Hoshiai and Mr. M. Takahashi¹ that ultrashort electric waves can be generated in a vacuum tube of special form by ordinary electric discharge. They observed the presence of extremely short wave of the order of 100 μ by means of Lecher wires. The bridging of light and electric waves was thus effected in a simple manner. This startling discovery was simply communicated as an experimental fact, without inquiring into the mechanism how the wave is generated. Subsequent experiment by Mr. T. Mishima in my laboratory indicated the presence of short waves in ordinary vacuum discharge at various pressures, though very short waves as given by Hoshiai and Takahashi remain as yet unconfirmed. Rough sketch of a theory is necessary for further development of the experiment by this process of generating ultrashort waves.

The theory which is closely connected with the present problem was given by L. Tonks and I. Langmuir² in a paper on the oscillations in ionized gases; J. J. Thomson³ treated the problem in one dimension, and gave results which are applicable to the present problem. There is, however, one lack in the assumption that the number of electrons and positive ions is expressible as oscillating functions. It is only a fraction of these numbers that can be so assumed, expressing the condensations and rarefactions similar to aerial vibrations, since the negative number has no physical meaning in the present case.

Consider a straight vacuum tube with its axis in x-direction with centres of anode and cathode in the same line; then the motion of electrons and ions can be treated as depending only on x and time t. Near the wall of the tube, the motion is complicated so that one-dimensional treatment is inapplicable to such places and must be excluded. Let \( n_0 \) and \( n'_0 \) denote the mean number per unit volume of electrons and positive ions resp. at x. As they are fluctuating, let the number at time t be given by \( n = n_0(1+s) \) and \( n' = n'_0(1+s') \) resp., s and s' being the measure of fluctuations. During the discharge, the potential gradient in unstriated positive column depends on the diameter of the tube, pressure, nature of the gas, and the current flowing through it. Considering these factors to remain constant, the potential gradient in the column does not change in the unstriated state, so that the

³ J. J. Thomson: Conduction of Electricity through Gases, 2 (1933), 353.
relation \( \frac{\partial E}{\partial x} = 0 \) holds, \( E \) denoting the field in the direction of the tube. Probe experiments indicate that \( n' \) is greater than \( n \), though the number is hundred times smaller than in the negative glow. If the charge of an electron be \(-e\), Poisson’s equation becomes \( \frac{\partial E}{\partial x} = 4\pi(n' - n)e \), which is a small quantity practically negligible. Consequently, the result here obtained is an approximation to the order of magnitude above cited. In the negative glow, the numbers of electrons and positive electrons are much more numerous than in the positive column, but \( n - n' \) is small compared with either \( n \) or \( n' \). Consequently \( \frac{\partial E}{\partial x} = 4\pi(n's' - n'\delta)e \).

Ordinary vacuum tube is bounded with glass wall, on which electrons and sometimes ions accumulate during the discharge, producing electric field in the neighbourhood. The fluctuations \( s \) and \( s' \) are affected by it, so that they are not uniform in any section normal to the axis of the tube. This is well exemplified by Pupp’s experiment\(^1\), in which \( s \) is great along the axis and small near the walls. Such minute consideration is difficult to enter in one dimensional problem, so that the boundary condition is outside the discussion. The fluctuations near the electrodes, the Crookes and Faraday dark spaces, and the cathode fall is difficult to enter, as the measurements of the field with probe are also affected with errors, which cannot be easily eliminated, unless some other method of sounding without disturbing the field is developed.

With the above approximation, the fluctuations \( s \) and \( s' \) can be found with the aid of gas theory, which is still applicable to pressures of a few mm., but fails in high vacuum. The equation for transference in \( x \)-direction of a physical quantity \( Q \) is given by

\[
\frac{d}{dt} (\bar{Q}) = -\frac{\partial}{\partial x} (n\bar{u}Q) + \frac{n}{m} X \frac{\partial \bar{Q}}{\partial u} + \lambda Q;
\]

\( \bar{Q} \) is the mean value of \( Q \), and \( X \) force per unit volume, and \( \bar{u} \) and \( \bar{u}' \) denote the velocity of electrons and positive ions resp., and the mean value is denoted by \( \bar{u} \) and \( \bar{u}' \). The equation of continuity is given by \( \frac{\partial n}{\partial t} + \frac{\partial}{\partial x} (n\bar{u}) = 0 \), when \( \bar{Q} = \text{const} \). This is equivalent to

\[
\frac{\partial \bar{s}}{\partial t} + \bar{u} \frac{\partial \bar{s}}{\partial x} = 0 \tag{1}
\]

or

\[
\frac{\partial \bar{s}'}{\partial t} + \bar{u}' \frac{\partial \bar{s}'}{\partial u} = 0 \tag{1'}
\]

As there are continuously collisions between electrons, ions, atoms and molecules, \( y \) and \( z \) components of velocity have values, which must be strictly taken into account, but to the degree of approximation above cited, they can be neglected.

The momentum communicated per second to the electrons in unit volume by the impact of positive ions is \( L = l \cdot n'(\bar{u}' - \bar{u}) \)

According to the gas theory $u = u + a$, $u' = u' + a'$, where $a$, $a'$ are Maxwellian velocities, so that $a = 0$, $a' = 0$; moreover $\bar{u} ^2 \ll a ^2$ and $\bar{u}' ^2 \ll a' ^2$, giving $\bar{u} ^2 = a ^2 = a$ and $\bar{u}' ^2 = a' ^2 = a'$, where $a$ and $a'$ are put for brevity.

The equation of transference of momentum for electrons becomes

$$m \frac{\partial}{\partial t} (\bar{u} \bar{s}) + m \frac{\partial}{\partial x} (\bar{n} \bar{u}^2) = -Een + L,$$

or

$$m \bar{u} \frac{\partial s}{\partial t} + m \bar{n} \bar{u} \frac{\partial s}{\partial x} = -Een + L.$$

Differentiating with respect to $x$

$$m \bar{u} \frac{\partial^2 s}{\partial t \partial x} + m \bar{n} \bar{u} \frac{\partial^2 s}{\partial x^2} = -en \frac{\partial E}{\partial x} + \frac{\partial L}{\partial x}$$

Since $\frac{\partial L}{\partial x} = l \cdot n_0 n_0' (\bar{u}' - \bar{u}) \left( \frac{1}{\bar{u}} \frac{\partial s}{\partial t} + \frac{1}{\bar{u}'} \frac{\partial s'}{\partial t} \right)$, it is equivalent to a damping term, and can be neglected for approximate calculation. In the negative glow, $\frac{\partial E}{\partial x} = 4\pi e(n_0' s' - n_0 s)$, so that

$$\frac{\partial^2 s}{\partial t^2} - a \frac{\partial^2 s}{\partial x^2} + \gamma (n_0 s - n_0' s') = 0$$

and

$$\frac{\partial^2 s'}{\partial t^2} - a' \frac{\partial^2 s'}{\partial x^2} - \beta (n_0 s - n_0' s') = 0$$

where $\gamma = 4\pi e^2/m$ and $\beta' = 4\pi e^2/m'$. In the unstriated positive column, $\frac{\partial E}{\partial x} = 0$, and

$$\frac{\partial^2 s}{\partial t^2} - a \frac{\partial^2 s}{\partial x^2} = 0$$

and

$$\frac{\partial^2 s'}{\partial t^2} - a' \frac{\partial^2 s'}{\partial x^2} = 0$$

Equations (2), (2'), (3), (3') are of vibrational form; the last two are of the same form as for vibrations in acoustics, considering $s$, $s'$ to be variations of density. Thus in the positive column, the ionic waves travel with velocities $\sqrt{\alpha}$ and $\sqrt{\alpha'}$, the former being much greater than the latter, as electrons move more swiftly than positive ions. With ordinary diatomic gases, the state of ionization is more complex than it has been assumed at the outset, since there may be atomic and molecular ions mixed together, though the former is greater in number than the latter in the ordinary vacuum discharge.

For solving (2) and (2'), suppose $s$ and $s'$ are oscillatory, of the form $e^{i(pot+q)}$, where $p = 2\pi V/\lambda$ and $q = 2\pi/\lambda$, and $V$ denotes the phase velocity and $\lambda$ the wave-length. By substitution, (2) and (2') becomes
\[(p^2 - aq^2 - \gamma n_0)s - \gamma n_0's = 0\]  
\[(p^2 - aq^2 - \gamma' n_0)s' - \gamma n_0s = 0\]

whence  
\[(p^2 - aq^2 - \gamma n_0)(p^2 - aq^2 - \gamma' n_0') = \gamma' \gamma n_0n_0'\]

Since \(\gamma' / \gamma = m / m'\) and \(a'/a = a'^2/a^2\) are very small and \(n_0 \approx n_0'\), the roots are

\[p_1^2 \approx aq^2 + \gamma n_0\]
\[p_2^2 \approx a'q^2 + \frac{\gamma' n_0q^2}{aq^2 + \gamma n_0}\]

Since the mean velocity square of electrons is much greater than that of positive ions, (5) indicates that \(p_1 \gg p_2\).

The relative amplitudes of electronic and ionic vibrations are found by (2) and (2') for \(p_1\) and \(p_2\) to be approximately

\[s_1/s_1' \approx n_0q^2/\gamma'\] and \[s_2/s_2' \approx -n_0q^2/\gamma\]

These equations indicate that the condensation and rarefaction of electrons and positive ions are in the same phase when the wave travels with velocity \(V_1\), corresponding to \(p_1\), and in the opposite phase with velocity \(V_2\) given by \(p_2\). The electron density in both \(p_1\)- and \(p_2\)-waves shows great variation as compared with ion density. Since \(s_1s_2 = -n_0m'/n_0m\) is nearly the ratio of the atomic weight of the gas to that of an electron, the electrons play the principal part in generating electric waves in a vacuum tube. Perhaps the ionic condensation has important connection with luminosity of the positive column.

Reverting to equation (5), the velocity \(V_1\) is given by

\[V_1^2 = \alpha + \frac{e^2n_0\alpha^2}{\pi m}\]

and

\[V_2^2 = \alpha' + \frac{mn_0e^2\alpha^2}{m'\pi am + n_0e^2\lambda^2}\]

Taking \(V_1\) and \(\lambda\) as abscissa and ordinate resp., the dispersion curve for (7) becomes a hyperbola \(H\) as in the figure, with semiaxes \(\sqrt{\alpha}\) and \(\sqrt{\frac{\pi ma}{e^2n_0}}\), indicating increase of \(V_1\) with \(\lambda\); and for long waves, \(V_1\) is nearly proportional to \(\lambda\), as the curve approaches the asymptote \(V_1 = \sqrt{\frac{e^2n_0\lambda}{\pi m}}\). With increasing wave-length, the frequency \(\frac{V_1}{\lambda} = \frac{p_1}{2\pi}\) reaches a limiting value \(\sqrt{\frac{e^2n_0}{\pi am}}\). The frequency cannot be smaller than the above value, so that the oscil-
lating electrons or ions give rise ultimately to ultrashort electromagnetic waves.

The relation between $V_2, \lambda$ is represented by a quartic curve. Only when $n_0 e^2 \lambda^2 / \pi m a$ is very small, can it be represented by

$$V_2^2 = a' + \frac{n_0 e^2 \lambda^2}{\pi m'}$$

with limiting frequency $\sqrt{\frac{e^2 n_0}{\pi m}}$, which is much smaller than for $V_1$. The case is only mathematical, as the above condition is hardly fulfilled in practice.

The group velocity $G = V - \frac{\partial V}{\partial \lambda}$, and $G_1 = a/V_1$, which is less than $\sqrt{\frac{\alpha}{\lambda}}$, since $V_1 > \sqrt{\alpha}$ and $G_2 = V_2 - \frac{n_0 m'}{e n_0 m a} \frac{V_2^2 - a'}{V^2 \lambda^2}$, which is less than $\sqrt{\frac{\alpha}{\lambda}}$, and equal to $V_2$ for infinite wavelength. The velocity of energy transmission is consequently slower than the square root of the mean velocity square of electrons or ions in these two cases.

The vibrations in the positive column, being determined by (3) and (3'), are independent of each other for electrons and positive ions, and of simple nature like sound vibrations in a wide long tube, travelling with phase velocity $\sqrt{\alpha}$ for electrons and $\sqrt{\alpha'}$ for ions. Since the vibrations here considered are of elastic type, it seems quite likely that they are transmitted from the negative glow to the Faraday dark space, and pass to the positive column, in which the electrons vibrate with the same frequency as in the negative glow and similarly for the ions. The transference of vibrations into the space containing only a small number of electrons and ions is somewhat questionable, but as the positive column can respond to these vibrations, the weak excitation through relatively short Faraday space will gradually give rise to vibrations of appreciable amplitude through positive column, which extends to the neighbourhood of the anode. The amplitude of vibration of electrons is much greater than that of positive ions, as the mass is many thousand times smaller in the former than in the latter.

An electrified particle vibrating in x-direction, with displacement $\xi = b \sin (pt + \epsilon)$, can be looked upon approximately as an oscillator, and the field produced about it consists of electric force $E_\theta$ and magnetic force $E_\phi$ given by

$$E_\theta = \frac{8}{\pi b \tau} \sum_{n=1}^{\infty} n J_n (n \beta \cos \theta) \sin n \left( p \left( t - \frac{\tau}{c} \right) + \epsilon \right)$$

$$E_\phi = - \frac{2}{\pi b \tau} \left( p \left( t - \frac{\tau}{c} \right) + \epsilon \right)$$

where $\theta$ denotes the angle of the vibrating axis with radius vector connecting the mean point of the electrified particle with 'aufpunkt', and $\phi$ longitude in the plane through the point perpendicular to the

1) Schott: Electromagnetic Radiation (1912), 120.
axis. $J_n$ denotes Bessel function of $n$-th order, $\beta = \frac{bp}{c}$. Thus each electron and positive ion in a state of vibration are sources of electromagnetic waves. Since the electron density is of order $10^8$ to $10^{11}$, the limiting wavelengths for vibrating electrons is of order of a few meters down to mms. for most simple gases. If we consider the existence of high harmonics, whose amplitudes are however very small, a great part of Hoshiai and Takahashi’s experiment can be accounted for. The special form of the tube used by them requires further experimental as well as theoretical inquiry.

Hitherto there was difficulty in interpreting the luminosity of the positive column, where the electron density is hundred times smaller than in the negative glow. Assuming the vibrations of electrons in the column as treated above, the ultrashort wave generated by numerous electrons can excite the atoms and molecules, giving line and band spectra. Spectroscopists were of opinion that the electric field produced by the voltage difference at the electrodes was the cause of excitation of light, without taking account of electric waves that necessarily accompany the discharge. From the present treatment of the subject, it appears probable that the wave also plays an important part in producing high luminosity of the positive column. The moving striations well observed in inert gases seem also to be connected with the vibrations here treated.

An important practical problem is how to obtain ultrashort waves of desired frequency, which is higher than the limiting value. Hoshiai and Takahashi produced very short waves by introducing a wire bent in a special curve to each of the Lecher wires symmetrically. It would mean an insertion of a capacity and selfinductance, by which the frequency can be properly adjusted, but it awaits further experimental tests to explain the real meaning of this arrangement.