Let $E$ be a closed set of capacity zero\(^1\) on the $z$-plane and $C$ be a Jordan curve surrounding $E$ and $D$ be the domain bounded by $E$ and $C$. Let $w = w(z)$ be one-valued and meromorphic in $D$ and on $C$ and have an essential singularity at every point of $E$ and $F'$ be the Riemann surface of the inverse function $z = z(w)$ of $w = w(z)$ spread over the $w$-plane. Concerning $F'$, the following facts are known: (i) $F'$ covers any point on the $w$-plane infinitely many times, except a set of points of capacity zero\(^2\). (ii) Let $w_0$ be a regular point of $F'$. Then $z(w)$ can be continued analytically on the half-lines $w = w_0 + re^{i\theta}$ $(0 \leq r < \infty)$ indefinitely or till we meet the image of $C$, except a set of values of $\theta$ of measure zero\(^3\).

Let $(w_0)$ be a boundary point of $F'$, whose projection on the $w$-plane is $w_0$. Iversen called $(w_0)$ a direct transcendental singularity of $z(w)$, if $w_0$ is lacunary for a connected piece $F_0$ of $F$, which lies above a disc $K_0$ about $w_0$ and has $(w_0)$ as its boundary point.

We will prove the following third property of $F'$.

**Theorem.** The set of points on the $w$-plane, which are the projections of direct transcendental singularities of $z(w)$ is of capacity zero.

We will first prove a lemma.

**Lemma.** Let $F_0$ be a connected piece of a Riemann surface $F$ spread over the $w$-plane, which lies above a disc $K_0$ bounded by a circle $C_0$. Suppose that $F_0$ does not cover a closed set $E_0$ which lies with its boundary inside $C_0$. If there exists a non-constant $f(w)$ on $F_0$, which satisfies the following conditions: (i) $f(w)$ is one-valued and meromorphic on $F_0$, (ii) $f(w)$ does not take the values on a closed set $E$ of capacity zero, (iii) $f(w)$ tends to $E$, when $w$ tends to any accessible boundary point of $F_0$, whose projection lies inside $C_0$, then $\text{cap. } E_0 = 0$.

**Proof.** Let $\tilde{F}$ be the simply connected universal covering Riemann surface of $F_0$. We map $\tilde{F}$ on $|x| < 1$ by $w = \varphi(x)$. Suppose that $\text{cap. } E_0 > 0$, then, as I have proved in my former paper\(^4\), the accessible

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1) In this note, "capacity" means "logarithmic capacity."


4) M. Tsuji: On the domain of existence of an implicit function defined by an integral relation $G(x, y) = 0$. Proc. 19 (1943).
boundary points of $F_0$, whose projections lie inside $C_0$, correspond to a set $e_0$ of positive measure on $|x|=1$. Since $\text{cap. } E=0$, by Evans’ theorem, there exists a positive mass-distribution $d\mu(a)$ of total mass 1 on $E$, such that

$$u(z)=\int_E \log \frac{1}{|z-a|} d\mu(a)$$

tends to $+\infty$, when $z$ tends to any point of $E$. Let $v(z)$ be the conjugate harmonic function of $u(z)$ and put $H(z)=e^{-(u+i\phi)}$. Then $H(z)$ is meromorphic outside $E$ and tends to zero, when $z$ tends to any point of $E$. We put $G(z)=H(f(\varphi(x)))$. Then $G(z)$ is one-valued and meromorphic in $|x|<1$.

If $x$ tends to any point of $e_0$ non-tangentially to $|x|=1$, $f(\varphi(x))$ tends to $E$, so that $G(x)$ tends to zero. Hence by Priwaloff’s theorem, $G(x) \equiv 0$, or $f(w) \equiv \text{const.}$, which contradicts the hypothesis. Hence $\text{cap. } E_0=0$, q.e.d.

By this lemma, we can prove the Theorem as follows.

Let $F_0$ be a connected piece of $F$, which lies above a disc $K_0$ bounded by a circle $C_0$. We suppose that $F_0$ does not contain the image of $C$.

We see easily that, $z(w)$ tends to $E$, when $w$ tends to any accessible boundary point of $F_0$, whose projection lies inside $C_0$. Since $z(w)$ does not take the values on $E$, we have by the Lemma, that the set of points inside $C_0$ which are uncovered by $F_0$ is of capacity zero. This point established, we can proceed similarly as in my former paper and prove the Theorem.

6) M. Tsuji, l.c. 4).