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As the trouble in highly revolving turbines is very often caused by the vibration of the disc, extensive researches on this subject are being made both at home and abroad. Although the most powerful turbines are generally of marine use and accordingly partake the complicated motions of ships, yet so far no attention has been paid to the effect of these motions, which may in some cases be fatal to the turbines. The present note deals with the effect of the yawing or pitching of a turbine ship on the turbine discs which are thereby set to precessional motions.

In the diagram, let D be a turbine disc revolving with an angular velocity $\omega$ around the axis which is taken to be the axis of $z$. The vertical axis through the centre of the disc $O$ is taken as $y$ and the axis perpendicular to both axes as $x$. Let us suppose that the disc precesses around the axis of $y$ with an angular velocity $\chi$. Then an elementary mass $m$ of the disc at a position $r, \theta$ will be subjected to an acceleration perpendicular to the plane of the disc and of the amount

$$2r\omega\chi \sin \theta,$$

the force which thereby comes into play being generally called gyroscopic force.

Now, without materially upsetting the force of the argument, we may assume that the above is equivalent to the case where the disc is at
rest, while the gyroscopic force,

\[ 2m\rho \omega Z \sin (\theta + \omega t) \]

revolving in the opposite direction to the actual revolution of the disc acts on the disc. So far as the behaviour of the disc is concerned, both cases will mechanically be the same, excepting that the effect of the centrifugal stress is ignored in the hypothetical case.

The equation of the elastic vibration of the disc may then be written as

\[ \rho \frac{\partial^2 \omega}{\partial t^2} - n \rho \sin (\theta + \omega t) + \frac{Eh^2}{3(1-\sigma^2)} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)^2 \omega = 0, \]

in which

- \( w \) = displacement of neutral surface in z-direction,
- \( r, \theta \) = co-ordinates of point on neutral surface,
- \( \rho \) = density of material of plate,
- \( E \) = Young's modulus,
- \( \sigma \) = Poisson's ratio,
- \( 2h \) = thickness of disc,
- \( n = 2\omega Z \).

Therefore, the forced vibration is given by

\[ w = W \sin (\theta + \omega t), \]

where \( W \) is a function of \( r \) satisfying

\[ \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \right)^2 W - k^4 W = m^2 r \]

in which

\[ k^4 = \frac{3(1-\sigma^2)\omega^2 \rho}{Eh^2} \quad \text{and} \quad m^2 = \frac{3(1-\sigma^2)n \rho}{Eh^2}. \]

It can easily be shown that the general solution of (1) is

\[ W = AJ_1(kr) + BY_1(kr) + CI_1(kr) + DK_1(kr) - \frac{n}{\omega^2} r \]

\[ J_1, Y_1, I_1, K_1 \] being Bessel's functions and their modified forms all of the first order. Therefore, the required forced vibration is given by

\[ w = \left\{ AJ_1(kr) + BY_1(kr) + CI_1(kr) + DK_1(kr) - \frac{n}{\omega^2} r \right\} \sin (\theta + \omega t) \]

Now, the boundary conditions to be satisfied are as follows:

At the outer edge \((\text{radius}=a)\),

\[ \frac{\partial^2 w}{\partial r^2} + \rho \left( \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) = 0. \]

\[ \frac{\partial}{\partial r} \Delta^2 w + \frac{(1-\sigma)}{r^2} \frac{\partial^2}{\partial \theta^2} \left( \frac{\partial w}{\partial r} - \frac{w}{r} \right) = 0. \]
On the circumference of the shaft hole (radius = \( b \)),
\[
\begin{cases}
w = 0, \\
\frac{\partial w}{\partial r} = 0.
\end{cases}
\]

These conditions give the following simultaneous equations which are written in the order of the above conditions.
\[
\begin{align*}
A \{ J_1(ka) - aJ_2(ka) \} + B \{ Y_1(ka) + aY_2(ka) \} - C \{ I_1(ka) - aI_2(ka) \} - D \{ K_1(ka) + aK_2(ka) \} &= 0, \\
AJ_2(ka) + BY_2(ka) + CI_2(ka) - DK_2(ka) &= 0, \\
AJ_1(kb) + BY_1(kb) + CI_1(kb) + DK_1(kb) &= \frac{n}{\omega^2} b, \\
AJ_0(kb) + BY_0(kb) + CI_0(kb) - DK_0(kb) &= \frac{2n}{\omega^2},
\end{align*}
\]
in which \( a \) stands for \( \frac{1 - \sigma}{ak} \).

From these equations we obtain
\[
A = \begin{bmatrix}
\frac{n}{\omega^2} & Y_1(ka) - aY_2(ka) & -I_1(ka) + aI_2(ka) & -K_1(ka) - aK_2(ka) \\
0 & Y_2(ka) & I_2(ka) & -K_1(ka) \\
b & Y_1(kb) & I_1(kb) & K_1(kb) \\
\frac{2}{k} & Y_0(kb) & I_0(kb) & -K_0(kb)
\end{bmatrix}
\]
\[
B = \begin{bmatrix}
\frac{n}{\omega^2} & J_1(ka) - aJ_2(ka) & 0 & -I_1(ka) + aI_2(ka) & -K_1(ka) - aK_2(ka) \\
J_2(ka) & 0 & I_2(ka) & -K_1(ka) \\
J_1(kb) & b & I_1(kb) & K_1(kb) \\
J_0(kb) & \frac{2}{k} & I_0(kb) & -K_0(kb)
\end{bmatrix}
\]
\[
C = \begin{bmatrix}
\frac{n}{\omega^2} & J_1(ka) - aJ_2(ka) & Y_1(ka) - aY_2(ka) & 0 & -K_1(ka) - aK_2(ka) \\
J_2(ka) & Y_2(ka) & 0 & -K_2(ka) \\
J_1(kb) & Y_1(kb) & 6 & K_1(kb) \\
J_0(kb) & Y_0(kb) & \frac{2}{k} & -K_0(kb)
\end{bmatrix}
\]
where

\[ \Delta = \begin{vmatrix}
J_1(ka) - aJ_2(ka) & Y_1(ka) + aY_2(ka) & I_i(ka) + aI_d(ka) \\
J_2(ka) & Y_2(ka) & I_i(ka)
\end{vmatrix} \]

If the values of the coefficients thus found are substituted in (3), we have an equation representing the distorted form progressing with the speed \( \omega \) on the disc under the action of the gyroscopic force. As however, that which actually revolves is the disc and not the force, the deformation does not progress but remains stationary relative to a frame of reference following the precessional motion (namely the ship).
As an example, a revolving steel disc \((E = 2.14 \times 10^{12} \text{ dyne per sq. cm. and } \rho = 7.85)\), having an outer radius of 100 cm., an inner radius of 40 cm. and a thickness of 2 cm., is supposed to be fitted athwart on board a ship which pitches with an amplitude of 6 degrees and with a period of 6 seconds. The result of the calculations of the deflection of the free edge and the maximum radial bending stress at the inner edge of the disc is shown in the annexed diagram. Both the deflection and the maximum bending stress correspond to the maximum precessional velocity due to the pitching of the ship. It will be seen that the effect may be serious and deserves a keen attention.

So far, the effect of the centrifugal stress which affects the vibration of the disc has not been considered. It is well known that the frequency of the vibration of a disc with one nodal diameter, when it is revolving with a frequency \(\omega\), is approximately modified in a manner as expressed by the following relation:

\[
p_s^2 = p^2 + \omega^2
\]

\(p\) and \(p_s\) being the frequencies of vibration of the disc at rest and in motion respectively. Therefore, the actual behaviour will be somewhat different from that here considered; indeed, not only the phenomenon of resonance does not actually occur, but also the revolution of the disc never overruns its frequency of vibration. The modification of the solution required when the centrifugal stress is taken into consideration is now being studied by the author.