124. *Propagation of Rayleigh-Waves on a Spherical Surface.*

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The equations of motion of elastic bodies in spherical coordinates, when the azimuthal component of the motion is omitted, are given by

$$\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu) \frac{\partial \Delta}{\partial r} - \frac{2\mu}{r} \frac{\partial \varpi}{\partial \theta} - \frac{2\mu}{r} \varpi \cot \theta \quad (1)$$

$$\rho \frac{\partial^2 v}{\partial t^2} = \frac{\lambda + 2\mu}{r} \frac{\partial \Delta}{\partial \theta} + 2\mu \frac{\partial \varpi}{\partial r} + \frac{2\mu \varpi}{r} \quad (2)$$

where u, v = radial and colatitudinal components respectively.

$$\left. \begin{aligned} \Delta &= \frac{\partial u}{\partial r} + \frac{2u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{v}{r} \cot \theta \\ 2\varpi &= \frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r} \frac{\partial u}{\partial \theta} \end{aligned} \right\} \quad (3)$$

The elimination of u and v among (1), (2) and (3) gives

$$\frac{\partial^2 \Delta}{\partial t^2} = \frac{\lambda + 2\mu}{\rho} \left\{ \frac{\partial^2 \Delta}{\partial r^2} + \frac{2}{r} \frac{\partial \Delta}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Delta}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial \Delta}{\partial \theta} \cot \theta \right\} \quad (1')$$

$$\frac{\partial^2 \varpi}{\partial t^2} = \frac{\mu}{\rho} \left\{ \frac{\partial^2 \varpi}{\partial r^2} + \frac{2}{r} \frac{\partial \varpi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \varpi}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial \varpi}{\partial \theta} \cot \theta - \frac{\varpi}{r^2} (1 + \cot^2 \theta) \right\} \quad (2')$$

Writing $\Delta = \Delta_1 e^{i\nu t}$ and $\varpi = \varpi_1 e^{i\nu t}$ in (1') and (2') respectively and solving the resulting equations, we obtain,

$$\Delta_1 = A \frac{J_m(hr)}{\sqrt{r}} \left\{ P_n(\cos \theta) + A' Q_n(\cos \theta) \right\} \quad (4)$$

$$\varpi_1 = B \frac{J_m(kr)}{\sqrt{r}} \left\{ \frac{d P_n(\cos \theta)}{d \theta} + B' \frac{d Q_n(\cos \theta)}{d \theta} \right\} \quad (5)$$

in which $h^2 = \frac{\rho p^2}{\lambda + 2\mu}$, $k^2 = \frac{\rho p^2}{\mu}$, $m^2 = \frac{1}{4} + n(n+1)$, and n is a constant to denote the number of waves on the whole spherical surface.

Displacement (u_1, v_1) answering to A_1 in (4) and satisfying $\varpi=0$ is given by

$$u_1 = -\frac{A}{h^2} \frac{d}{dr} \frac{J_m(hr)}{\sqrt{r}} \left\{ P_n(\cos \theta) + A' Q_n(\cos \theta) \right\}$$

$$v_1 = -\frac{A}{h^2} \frac{J_m(hr)}{r^{\frac{3}{2}}} \left\{ \frac{d P_n(\cos \theta)}{d\theta} + A' \frac{d Q_n(\cos \theta)}{d\theta} \right\}$$

Displacement (u_2, v_2) derived from the value of ϖ_1 in (5) with the condition $A=0$ is expressed by

$$u_2 = -2B \frac{n(n+1)}{k^2} \frac{J_m(kr)}{r^{\frac{3}{2}}} \left\{ P_n(\cos \theta) + B' Q_n(\cos \theta) \right\}$$

$$v_2 = -2B \frac{1}{k^2} \frac{1}{r} \frac{d}{dr} \left\{ \sqrt{r} J_m(kr) \right\} \left\{ \frac{d P_n(\cos \theta)}{d\theta} + B' \frac{d Q_n(\cos \theta)}{d\theta} \right\}$$

The surface $r=a$ being free from traction, the equations

$$\lambda A + 2\mu \frac{\partial u}{\partial r} = 0 \quad \text{and} \quad \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r} \frac{\partial u}{\partial \theta} = 0 \quad (6)$$

in which $u=u_1+u_2$ and $v=v_1+v_2$, must hold on that surface.

Eliminating A and B by the application of the above boundary conditions at $r=a$, we obtain the following equation to determine the velocity of propagation of surface waves.

$$\begin{aligned} & (\lambda + 2\mu) \frac{h^2 J_m(ha)}{\sqrt{a}} \left[\left(-k^2 + \frac{2(n^2 + n - 1)}{a^2} \right) \frac{J_m(ka)}{\sqrt{a}} - \frac{2}{a} \left(\frac{d}{dr} \frac{J_m(kr)}{\sqrt{r}} \right)_{r=a} \right] \\ & - 2\mu \left[\frac{n(n+1)}{a^2} \left(-k^2 + \frac{2(n-1)(n+2)}{a^2} \right) \frac{J_m(ha)}{a} \right. \\ & \quad \left. - \frac{2}{a} \left(-k^2 + \frac{(n-1)(n+2)}{a^2} \right) \frac{J_m(ka)}{\sqrt{a}} \left(\frac{d}{dr} \frac{J_m(hr)}{\sqrt{r}} \right)_{r=a} \right. \\ & \quad \left. - \frac{2(n-1)(n+2)}{a^2} \left(\frac{d}{dr} \frac{J_m(hr)}{\sqrt{r}} \frac{d}{dr} \frac{J_m(kr)}{\sqrt{r}} \right)_{r=a} \right] = 0 \quad (7) \end{aligned}$$

This equation is too complicated to study the nature of the motion, so that we must be contented with the following simplified form:—

$$\left(\frac{k^2}{f^2} - 2 \right)^4 = 16 \left(1 - \frac{h^2}{f^2} \right) \left(1 - \frac{k^2}{f^2} \right) \quad (7')$$

where $\frac{h}{f} = \frac{ha}{n}$ and $\frac{k}{f} = \frac{ka}{n}$.

In this the largeness of n is taken into consideration; even when $n=100$, the wave length on the surface of the Earth is so large as 400 kilometers. Also for the values of the Bessel's functions in (7), the first term of the asymptotic expansion due to P. Debye, namely

$$J_n(x) = \frac{1}{H} e^{-x (\tau \cosh \tau - \sinh \tau)} \sum_{q=0}^{q=l} B_q(\tau) \frac{\Gamma\left(q + \frac{1}{2}\right)}{\left(\frac{x}{2} \sinh \tau\right)^{q+\frac{1}{2}}},$$

in which $\cosh \tau = \frac{n}{x} > 1$, is taken.

The equation (7') is of the same form as that derived by Lord Rayleigh to determine the velocity of propagation of plane surface waves.

The velocity of propagation excepting near the origin, when n is large, is given by

$$\frac{2\pi/f}{2\pi/p} = C \sqrt{\frac{\mu}{\rho}}$$

where $C=0.9553$ for incompressible materials and 0.9194 for materials whose Poisson's ratio is $\frac{1}{4}$.

The value of B/A is obtained from one of the relations in (6). If the second relation is used,

$$\frac{B}{A} = -\frac{k^2}{h^2} \left\{ \frac{1}{r} \frac{d}{dr} \frac{J_m(hr)}{\sqrt{r}} - \frac{1}{r^2} \frac{J_m(hr)}{\sqrt{r}} \right. \\ \left. \frac{J_m(kr)}{\sqrt{r}} - \frac{2}{r} \frac{d}{dr} \frac{J_m(kr)}{\sqrt{r}} \right\}_{r=a} \quad (8)$$

Using a similar process by which we have deduced (7') from (7), we obtain

$$\frac{B}{A} = \frac{-\sqrt{1-\left(\frac{h}{f}\right)^2}}{n\left[2\left(\frac{h}{k}\right)^2-\left(\frac{h}{f}\right)^2\right]} \left\{ \frac{1-\left(\frac{k}{f}\right)^2}{1-\left(\frac{h}{f}\right)^2} \right\}^{\frac{1}{4}} \exp \left\{ -n \log \left[\frac{k}{h} \frac{1+\sqrt{1-\left(\frac{h}{f}\right)^2}}{1+\sqrt{1-\left(\frac{k}{f}\right)^2}} \right] \right. \\ \left. + n \left[\sqrt{1-\left(\frac{h}{f}\right)^2} - \sqrt{1-\left(\frac{k}{f}\right)^2} \right] \right\} \quad (8')$$

The value of B/A being thus found, we can arrive at the general expressions for displacements which can be written as follows:—

$$\left. \begin{aligned} u &= A \left[\frac{1}{h^2} \frac{d}{dr} \frac{J_m(hr)}{\sqrt{r}} + 2 \left(\frac{B}{A} \right) \frac{n(n+1)}{k^2} \frac{J_m(kr)}{r^{\frac{3}{2}}} \right] \times \\ &\quad \left[P_n(\cos \theta) \sin pt + D_n Q_n(\cos \theta) \cos pt \right] \\ v &= A \left[\frac{1}{h^2} \frac{J_m(hr)}{r^{\frac{3}{2}}} + \frac{2}{k^2} \left(\frac{B}{A} \right) \frac{1}{r} \frac{d}{dr} \left\{ \sqrt{r} J_m(kr) \right\} \right] \times \\ &\quad \left[\frac{d P_n(\cos \theta)}{d\theta} \sin pt + D_n \frac{d Q_n(\cos \theta)}{d\theta} \cos pt \right] \end{aligned} \right\} \quad (9)$$

where D_n is a positive number, whose value is constant for a given n and which is to be so adjusted that $P_n(\cos \theta)$ and $D_n Q_n(\cos \theta)$ may have enveloping curves in common, excepting near the antipode as well as origin within a zenith distance corresponding to, say, $\frac{1}{4}$ wave length. Motions in these regions must be excluded from the consideration.

The expressions (9) well indicate the fact that, while the displacements gradually decrease as the waves approach the equatorial circle, the displacements again increase as they proceed towards the antipode.