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Let \( \{ \varphi_n(x) \} \) be a system of normalized orthogonal functions in \((a, b)\) and consider the series

\[
\sum_{n=0}^{\infty} c_n \varphi_n(x)
\]

such that

\[
\sum_{n=0}^{\infty} c_n^2 < \infty.
\]

By the Riesz-Fisher theorem, the series (1) converges in the mean to a function \( f(x) \) in \( L^2 \). As usual we denote by \( s_n(x) \) and \( \sigma_n(x) \) the partial sum and \((C,1)\)-mean of the series (1) respectively. In this paper we discuss the convergency of

\[
\sum_{n=1}^{\infty} |s_n(x) - f(x)|^k / n, \quad k > 1.
\]

For the case of trigonometrical system, the former is considered by Hardy and Littlewood\(^1\) and the latter by Zygmund\(^2\).

As an application of our theory, we shall give an alternative proof of the Rademacher\(^3\)-Menchof\(^4\) theorem regarding the almost everywhere convergence of the series (1).

1. Convergency of the series \( \sum_{n=1}^{\infty} (s_n - f)^2 / n \).

\[(1.1) \quad \int_a^b \left( \sum_{n=1}^{\infty} (s_n - f)^2 / n \right) dx \leq A \sum_{n=1}^{\infty} c_n^d \log n^5.\]

For,

\[
\sum_{n=1}^{\infty} \frac{1}{n} \int_a^b (s_n - f)^2 dx = \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{\nu=1}^{\infty} c_{n+\nu}^2 \right) = \sum_{\nu=1}^{\infty} c_{\nu+1}^2 - \frac{1}{n} \leq A \sum_{n=1}^{\infty} c_n^d \log \nu,
\]

which is the required.

For the case of trigonometrical system, we have

\[
\sum_{n=1}^{\infty} c_n^d \log n \sim \int_{-\pi}^{\pi} \frac{|f(x+t)+f(x-t)-2f(x)|^2}{2t} dx dt.
\]

5) \( A, B, \ldots \) are constants, not always the same from one occurrence to another.
Thus (1.1) is a special case of the result due to Hardy and Littlewood.

(1.2) If \( \sum_{n=1}^{\infty} c_n \log n < \infty \), then \( \sum_{n=0}^{\infty} c_n \varphi_n \) is \((C, 1)\)-summable almost everywhere in \((a, b)\).

This is evident from (1.1) by the Kronecker theorem. This theorem is a classical result due to Weyl and Hobson. Moreover, Borgen proved that \( \sum_{n=1}^{\infty} c_n (\log (\log n))^2 < \infty \) is sufficient for almost everywhere \((C, 1)\)-summability. On the other hand, Chen proved that this is equivalent to the Rademacher-Menchof theorem in the next section.


(2.1) If \( \sum_{n=1}^{\infty} c_n (\log n)^2 < \infty \). Then \( \sum_{n=0}^{\infty} c_n \varphi_n \) is convergent almost everywhere in \((a, b)\).

The method of reduction of (2.1) from (1.2) has been sketched by Zygmund. But for the sake of completeness we reproduce it with some simplification.

If we put

\[
A_n = \left( \frac{n+a}{a} \right) \sim \frac{n^a}{(n+1)^a}, \quad S_n^a = \sum_{\nu=0}^{n} A_n, \quad \sigma_n^a = S_n^a / A_n^a,
\]

then we have

\[
\sigma_n^a - \sigma_{n-1}^a = \frac{\sum_{\nu=0}^{n} \nu A_n^a \varphi_n}{\nu A_n^a}
\]

and

\[
\int_a^b (\sigma_n^a - \sigma_{n-1}^a)^2 dx = \frac{\sum_{\nu=0}^{n} \nu^2 (A_n^a)^2 \varphi_n^2}{\nu^2 (A_n^a)^2 (n+1)}.
\]

Accordingly

\[
\sum_{n=1}^{\infty} \int_a^b (\sigma_n^a - \sigma_{n-1}^a)^2 dx = \sum_{n=1}^{\infty} \frac{\sum_{\nu=0}^{n} \nu^2 (A_n^a)^2 \varphi_n^2}{\nu^2 (n+1)}
\]

\[
\leq B \sum_{n=0}^{\infty} \frac{\sum_{\nu=0}^{n} \nu^2 (n+1)^{2a-1} \varphi_n^2}{(n+1)^{2a+1}}
\]

\[
\leq B \sum_{\nu=0}^{\infty} \nu^2 \sum_{n=\nu}^{\infty} (n+1)^{2a-1} / (n+1)^{2a+1}
\]

\[
\leq B \sum_{n=0}^{\infty} \nu^2 \sum_{n=\nu+1}^{\infty} (n+1)^{2a-1} / (n+1)^{2a+1}
\]

\[
\leq P+Q, \quad \text{say}.
\]

1) G. H. Hardy and J. E. Littlewood, loc. cit.
Then
\[ P \leq B \sum_{\nu=0}^{\infty} \nu^{2a} (\nu+1)^{-2a-1} \sum_{n=\nu}^{\infty} (n-\nu+1)^{2a-1} (1 \geq a > 1/2) \]
\[ \leq B \sum_{\nu=0}^{\infty} \nu^{-2a+1} \nu^{2a-2+1} \leq C_1 \sum_{\nu=1}^{\infty} \nu^2, \]
\[ Q \leq B \sum_{\nu=0}^{\infty} \nu^{-2a+1} \nu^{2a-2+1} \sum_{n=\nu+1}^{\infty} (n^{2a-1} \leq C_2 \sum_{\nu=1}^{\infty} \nu^2. \]

Thus we get
\[ \int_{a}^{b} \sum_{n=0}^{\infty} (\sigma_{n-1}^2 - \sigma_n^2)/(n+1) dx \leq D \sum_{n=0}^{\infty} c_n^2, \text{ where } 1 \geq a > 1/2. \]

In the analogous way, we get for \( a=1/2 \)
\[ \int_{a}^{b} \sum_{n=0}^{\infty} (\sigma_{n-1}^2 - \sigma_n^2)/(n+1) dx \leq E \sum_{n=1}^{\infty} c_n^2 \log n. \]

Thus we proved the theorem:

(2.1.1.) If \( 1 \geq a > 1/2 \) then we have
\[ \int_{a}^{b} \sum_{n=0}^{\infty} (\sigma_{n-1}^2 - \sigma_n^2)/(n+1) dx \leq A \sum_{n=0}^{\infty} c_n^2, \]
and for \( a=1/2 \),
\[ \int_{a}^{b} \sum_{n=0}^{\infty} (\sigma_{n-1}^2 - \sigma_n^2)/(n+1) dx \leq B \sum_{n=1}^{\infty} c_n^2 \log n. \]

Further we have

(2.1.2.) If \( \sum_{n=0}^{\infty} (\sigma_n^2)/(n+1) \rightarrow 0 \), then \( \sigma_{n+1/2+}^2 \rightarrow 0 \), for \( a > -1/2, \varepsilon > 0 \)
and \( s_n = o(\sqrt{n \log n}) \) for \( a = -1/2. \)

For, \( |S_n^{1/2+}| = \left| \sum_{\nu=0}^{n} S_\nu A_{n-1/2+}^2 \right| = \sum_{\nu=0}^{n} |\sigma_\nu A_{n-1/2+}^2 A_{n-1/2+}^*| \]
\[ \leq \sqrt{\sum_{\nu=0}^{n} (\sigma_\nu^2) A_{n-1/2+}^2 A_{n-1/2+}^*} \leq o(\nu^{n}) O(\sqrt{\sum_{\nu=0}^{n} A_{n-1/2+}^2 A_{n-1/2+}^*}) \]
\[ = o(\nu^{n}) O(\nu^{n^{2a+1/2+}}) = o(n^{2a+1/2+}). \]

The remaining part is analogous.

**Proof of the theorem.** If \( \sum_{n=1}^{\infty} c_n^2 \log n < \infty \), then by (1.2), (1) is \((C, 1)\)-summable. From (2.1.1.) and (2.1.2.), \( s_n = o(\sqrt{n \log n}) \). By the well known theorem, the series
\[ \sum_{n=1}^{\infty} c_n (n \log n)^{1/2} \]
converges almost everywhere, provided that \( \sum_{n=1}^{\infty} c_n^2 \log n < \infty \). Thus \( \sum_{n=0}^{\infty} c_n n \log n \) converges almost everywhere, provided that \( \sum_{n=1}^{\infty} c_n^2 (\log n)^2 < \infty \).

3. **Behaviour of the series** \( \sum_{n=1}^{\infty} |s_n - f|^4/n \).
(3.1) If \( |\varphi_n(x)| \leq K \) (n = 0, 1, 2, ...) and \( f \sim \sum_{n=0}^{\infty} c_n \varphi_n \), then
\[
\int_a^b \sum_{n=1}^{\infty} |s_n - f|^q \, dx \leq A \sum_{n=1}^{\infty} |c_n|^{q \nu^{-2}} \log n,
\]
and
\[
\left( \int_a^b \sum_{n=1}^{\infty} |s_n - f|^q \, dx \right)^{1/q} \leq B \left( \sum_{n=1}^{\infty} |c_n|^p (\log n)^{p/q} \right)^{1/p},
\]
where \( 1 < p \leq 2 \leq q < \infty \), \( 1/p + 1/q = 1 \).

For, by Paley's theorem,
\[
\sum_{n=1}^{\infty} \frac{1}{n} \int_a^b |s_n - f|^q \, dx \leq A \sum_{n=1}^{\infty} \frac{1}{n} \sum_{n=1}^{\infty} |c_n|^q \nu^{q-2}.
\]
The right hand side series is less than
\[
\leq A \sum_{n=1}^{\infty} |c_n|^q \nu^{q-2} \sum_{n=1}^{\nu^{-1}} 1/n \leq B \sum_{n=1}^{\infty} |c_n|^q \nu^{q-2} \log \nu.
\]
Further
\[
\left\{ \sum_{n=1}^{\infty} |c_n|^q \nu^{q-2} (\log \nu)^p \right\}^{1/q} \leq C \left( \sum_{n=1}^{\infty} |c_n|^p (\log \nu)^{p/q} \right)^{1/p}.
\]
Accordingly
\[
\left( \int_a^b \sum_{n=1}^{\infty} |s_n - f|^q \, dx \right)^{1/q} \leq C \left( \sum_{n=1}^{\infty} |c_n|^p (\log \nu)^{p/q} \right)^{1/p}.
\]
Thus we get the theorem.

Analogously we get

(3.2) If \( |\varphi_n(x)| \leq K \) (n = 0, 1, 2, ...), then
\[
\int_a^b \sum_{n=1}^{\infty} |s_n - f|^q \, dx \geq D \sum_{n=1}^{\infty} |c_n|^p n^{p-2} \log n
\]
and
\[
\left( \int_a^b \sum_{n=1}^{\infty} |s_n - f|^q \, dx \right)^{1/q} \geq E \left( \sum_{n=1}^{\infty} |c_n|^q (\log n)^{\nu/q} \right)^{1/q},
\]
where \( 1 < p \leq 2 \leq q < \infty \), \( 1/p + 1/q = 1 \).

These results were given by Izumi and Kawata\(^1\) under more severe conditions.

4. Behaviour of \( \sum_{n=1}^{\infty} |s_n - \sigma_n|^q/n \).

(4.1) If \( |\varphi_n(x)| \leq K \) (n = 0, 1, 2, ...), then
\[
\int_a^b \sum_{n=1}^{\infty} |s_n - \sigma_n|^q \, dx \leq A \sum_{n=1}^{\infty} n^{q-2} \, |c_n|^q,
\]
and
\[
\left( \int_a^b \sum_{n=1}^{\infty} |s_n - \sigma_n|^q \, dx \right)^{1/q} \leq B \left( \sum_{n=1}^{\infty} |c_n|^p \right)^{1/p},
\]
where \( 1 < p \leq 2 \leq q < \infty \), \( 1/p + 1/q = 1 \).

For,
\[
s_n - \sigma_n \sim \sum_{k=1}^{n} [c_k \varphi_k],
\]

By Paley's theorem,
\[
\int_a^b s_n - \sigma_n \, |^q \, dx = \int_a^b \left( \sum_{n=1}^n |v_n \varphi_n| \right)/(n+1) \, |^q \, dx \leq A \left( \sum_{n=1}^\infty |v_n \varphi_n \varphi^{q-2}| \right)/n^q.
\]
Therefore
\[
\sum_{n=1}^\infty \frac{1}{n} \int_a^b s_n - \sigma_n \, |^q \, dx \leq A \sum_{n=1}^\infty \frac{1}{n^q+1} \left( \sum_{n=1}^\infty |v_n \varphi_n \varphi^{q-2}| \right)
\]
\[
\leq A \sum_{n=1}^\infty \varphi^{q-2} | c_n |^q \sum_{n=1}^\infty 1/n^{q+1}
\]
\[
\leq B \sum_{n=1}^\infty \varphi^{q-2} | c_n |^q \cdot \varphi^{-q} \leq B \sum_{n=1}^\infty \varphi^{q-2} | c_n |^q.
\]
Thus we get the first inequality of the (4.1). The remaining is given by
\[
(\sum_{n=1}^\infty \varphi^{q-2} | c_n |^q)^{1/q} \leq C \left( \sum_{n=1}^\infty |c_n|^p \right)^{1/p}.
\]
Thus we complete the proof of theorem.

Analogously we get
\[
(\sum_{n=1}^\infty \varphi^{p-2} | c_n |^p)^{1/p} \leq D \left( \sum_{n=1}^\infty |c_n|^q \right)^{1/q}.
\]

These results were considered also by Izumi and Kawata\(^1\) under more severe conditions.

5. Behaviour of the sequence \(\{s_{p_r}\}\).

For any increasing sequence \(\{p_r\}\), \((C,1)\)-summability of \(\{s_{p_r}\}\) is considered by Zalcwasser\(^2\). He opened the problem: For any \(f(x) \in L^2\), is the sequence \(\{s_{p_r}\}\) \((C,1)\)-summable for all \(\{p_r\}\), where \(\varphi(x)\) is trigonometrical system. Regarding this problem we get

\[
(5.1) \text{If} \quad \sum_{r=1}^\infty c_r^2 \text{ converges,} \quad \text{(C,1)\-summability of} \quad \{s_{p_r}\} \quad \text{is equivalent to the convergency of} \quad \{s_{p_r}^2\}.
\]

\[
\text{If} \quad c_r \equiv 0, \text{ then we put} \quad \varphi_r(x) = (c_{p_r} \varphi_{p_r-1} + \cdots + c_{p_r} \varphi_p)/\gamma_r,
\]
where
\[
\gamma_r = (c_{p_r}^2 \varphi_{p_r-1} + \cdots + c_{p_r}^2)^{1/2},
\]

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2) Z. Zalcwasser, Studia Math., 6 (1936), pp. 82-88.
and consider the series \( \sum_{\nu=0}^{\infty} \gamma_{\nu} \psi_{\nu}(x) \).

Since \( \{\psi_{n}(x)\} \) is a normalized orthogonal system and \( \sum_{\nu=0}^{\infty} \gamma_{\nu}^{2} < \infty \), the \((C,1)\)-summability of \( \sum_{\nu=0}^{\infty} \gamma_{\nu} \psi_{\nu}(x) \) is equivalent to the convergency of \( \{t_{\nu}\} \) where \( t_{\nu} \) is the \( \nu \)-th partial sum of \( \sum_{\nu=0}^{\infty} \gamma_{\nu} \psi_{\nu}(x) \). Thus we get the theorem.

From this, Zalcwasser's problem will perhaps be negatively answered, but the author could not conclude it.