The object of this paper is to extend Vivanti’s theorem and its generalizations to functions defined by Dirichlet’s series.

1. Let $r_1, r_2, r_3, \ldots$ be a sequence of real numbers such that

$$0 < r_1 < r_2 < r_3 < \ldots \quad , \quad \frac{r_v}{v} \to \infty.$$ 

Then the integral function

$$G(z) = \prod_{v=1}^{\infty} \left(1 - \frac{z^2}{r_v^2}\right)^2$$

is of order 1 and of minimal type. Let us next consider the Dirichlet’s series:

$$D(s) = \sum_{v=1}^{\infty} c_{\gamma_v} e^{-\lambda_v s} \quad \left(0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \ldots \quad , \quad \lambda_v \to \infty\right).$$

Then we have

**Lemma 1.** The Dirichlet’s series

$$H(s) = \sum_{v=1}^{\infty} c_{\gamma_v} G(\lambda_v) e^{-\lambda_v s}$$

and (1.2) have the same convergence abscissa, when

$$\lim_{v=\infty} (\lambda_v - \lambda_{v-1}) \quad , \quad \lim_{\nu, v=\infty} (r_\nu - r_v) > 0.$$ 

After Dr. Cramér (1.3) converges at least in the domain, where (1.2) is convergent. So it suffices to prove the converse. To this purpose we will first calculate the order of $G(\lambda_\nu)$.

Let $n$ be an integer such that $r_n < \lambda_\mu < r_{n+1}$. By (1.4) we have then $r_v - r_{v-1} > h$, $r_\nu - \lambda_v > h$ for all $v$ and $\nu$. In general we can suppose that $h=1$.

Now

$$\frac{1}{G(\lambda_v)} \leq \prod_{v=1}^{\nu} \frac{1}{(\lambda_v - \lambda_{v-1})^2} \prod_{\nu=n+1}^{\infty} \left(1 + \frac{\lambda_v^2}{(r_\nu + \lambda_\mu)(r_\nu - \lambda_\mu)}\right)^2$$

$$\leq \frac{\lambda_v^{2n}}{(n!)^2} \prod_{\nu=n+1}^{\infty} \left(1 + \frac{\lambda_\mu^2 \epsilon_\nu}{\nu(\nu-n)}\right)^2 \quad \left(\epsilon_\nu = \frac{\nu}{r_\nu} < \epsilon^2\right)$$

2) See Carlson u. Landau, Göttiniger Nachrichten, 1921.
Suppose that (1.3) is convergent for \( \sigma > l \), then

\[
A_v = \sum_{\nu=1}^{v} c_{\nu} G(\lambda_{\nu}) = O\left(e^{\lambda_{\nu}(l+\epsilon)}\right).
\]

And

\[
\left| \sum_{\nu=1}^{n} c_{\nu} \right| < \sum_{\nu=1}^{n} c_{\nu} G(\lambda_{\nu}) \frac{1}{G(\lambda_{\nu})} \leq \left| \sum_{\nu=1}^{n-1} A_{\nu} \left( \frac{1}{G(\lambda_{\nu})} - \frac{1}{G(\lambda_{\nu+1})} \right) + \frac{A_n}{G(\lambda_n)} \right| < \text{Max}_{1 \leq \nu < n} |A_{\nu}| \cdot \sum_{\nu=1}^{n-1} \left| \frac{1}{G(\lambda_{\nu})} - \frac{1}{G(\lambda_{\nu+1})} \right| + \frac{|A_n|}{G(\lambda_n)},
\]

where

\[
\sum_{\nu=1}^{n-1} \left| \frac{1}{G(\lambda_{\nu})} - \frac{1}{G(\lambda_{\nu+1})} \right| = \sum_{\nu=1}^{n-1} \frac{|G(\lambda_{\nu+1}) - G(\lambda_{\nu})|}{G(\lambda_{\nu})G(\lambda_{\nu+1})} < e^{2\lambda_{\nu}} \sum_{\nu=1}^{n-1} |G(\lambda_{\nu+1}) - G(\lambda_{\nu})| < e^{2\lambda_{\nu}} \int_{0}^{\gamma_{n}} |G'(x)| \, dx < e^{4\lambda_{\nu}},
\]

so that

\[
\sum_{\nu=1}^{n} c_{\nu} = O\left(e^{\gamma_{n}(l+\epsilon')}\right).
\]

That is, (1.2) is convergent for \( \sigma > l \). q.e.d.

2. Consider the Dirichlet's series with real coefficients:

\[
f(s) = \sum_{\nu=1}^{\infty} a_{\nu} e^{-\gamma_{\nu} s},
\]

whose convergence abscissa is finite, for example \( \sigma = 0 \). From the sequence \((\lambda_{\nu})\) select a subsequence \((r_{\nu})\) such that

\[
\frac{r_{\nu}}{\nu} \to \infty \quad \text{and} \quad \lim_{\nu \to \infty} (r_{\nu} - r_{\nu-1}) \leq 0.
\]

Let \((\mu_{\nu})\) be the complementary sequence of \((r_{\nu})\), then we have

\[
f(s) = \sum_{\nu=1}^{\infty} a_{r_{\nu}} e^{-r_{\nu} s} + \sum_{\nu=1}^{\infty} a_{\mu_{\nu}} e^{-\mu_{\nu} s} = g(s) + h(s) \quad \text{say}.
\]

We will now distinguish two cases. First let the convergence abscissa of \( h(s) \) be greater than 0, then that of \( g(s) \) is 0. In this case the point \( s = 0 \) is a singular point of \( g(s) \), as the Carlson-Landau-Szász's theorem shows us, so that \( s = 0 \) is also a singular point of \( f(s) \). Next

let the convergence abscissa of \( h(s) \) be \( \sigma=0 \). By Lemma 1 the convergence abscissa of

\[
(2.2) \quad \sum_{\nu=1}^{\infty} a_{\nu \nu} G(\mu_{\nu}) e^{-\mu_{\nu} s} = \sum_{\nu=1}^{\infty} a_{\nu \nu} G(\lambda_{\nu}) e^{-\lambda_{\nu} s}
\]
is \( \sigma=0 \). If we suppose that \( a_{\nu \nu} \geq 0 \) for all \( \nu \), that is

\[
(2.3) \quad a_{\nu \nu} \geq 0
\]
with the exception of \( a_{r \nu} \), which is arbitrary, then we have

\[
(2.4) \quad a_{\nu \nu} G(\lambda_{\nu}) \geq 0
\]
for all \( \nu \). So by the Landau’s theorem\(^1\) \( s=0 \) is a singular point of (2.2). On the other hand Dr. Cramer\(^2\) proved that (2.2) has no singularities other than those of (2.1). It follows that \( s=0 \) is a singular point of (2.1). Thus we have established the following

**Lemma 2.** Let the Dirichlet’s series with real coefficients (2.1) have the finite convergence abscissa \( \sigma=\alpha \), and \( a_{\nu \nu} \geq 0 \) except \( (a_{r \nu}) \) which are real or complex, and

\[
\frac{r_{\nu}}{\nu} \rightarrow \infty, \quad \lim_{\nu \rightarrow \infty} (r_{\nu} - r_{\nu-1}), \quad \lim_{\nu, \lambda \rightarrow \infty} (r_{\nu} - \lambda_{\nu}) > 0.
\]

Then \( f(s) \) is singular at \( s=\alpha \).

This is a generalization of the Landau’s theorem.\(^3\)

3. Let us now proceed to our principal theorem. Take a general Dirichlet’s series (1.2), whose convergence abscissa is finite \( \sigma=\alpha \), and consider

\[
(3.1) \quad \sum_{\nu=1}^{\infty} a_{\nu \nu} e^{-\lambda_{\nu} s} \quad \text{and} \quad \sum_{\nu=1}^{\infty} b_{\nu \nu} e^{-\lambda_{\nu} s}
\]

Then at least one of (3.1) has the same convergence abscissa as (1.2). Let us suppose that \( a_{\nu \nu}, b_{\nu \nu} \geq 0 \). Then \( \sigma=\alpha \) is a singular point of at least one of (3.1), so that this point is also singular for (1.2)\(^4\). Thus we get the following

**Theorem 1.** Suppose that the Dirichlet’s series (1.2) has the finite convergence abscissa, \( \sigma=\alpha \), and \( 0 \leq \arg c_{\nu \nu} \leq \frac{\pi}{2} \) with the exception of \( c_{r \nu} \) such that

\[
\frac{r_{\nu}}{\nu} \rightarrow \infty, \quad \lim_{\nu \rightarrow \infty} (r_{\nu} - r_{\nu-1}), \quad \lim_{\nu, \lambda \rightarrow \infty} (r_{\nu} - \lambda_{\nu}) > 0.
\]

Then \( s=\alpha \) is a singular point of the function defined by (1.2).

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1) Cramer, loc. cit.
2) Landau, loc. cit.
3) Szasz, loc. cit.
4. Suppose that the conditions in the theorem are satisfied and that
\[ \lim_{n \to \infty} e^{2\pi i \nu_n \varphi} = e^{2\pi i \psi} \]
for some irrational number \( \varphi \). Let us consider the series
\[
\sum_{n=0}^{\infty} c_{\nu_n} G(\lambda_n) e^{2\pi i \nu_n \varphi} e^{-\gamma_n s} = \sum_{n=0}^{\infty} c_{\nu_n} G(\mu_n) e^{2\pi i \nu_n \varphi} e^{-\nu_n s},
\]
where \( p \) is a positive integer. Multiplying a constant we get
\[
- i \sum_{n=0}^{\infty} c_{\nu_n} G(\mu_n) e^{2\pi i \nu_n \varphi} e^{-\nu_n s} \left( \psi_n = \mu_n p \varphi - p \psi + \frac{1}{8} \right).
\]
At least one of the series
\[
\sum_{n=1}^{\infty} a_{\nu_n} G(\mu_n) e^{2\pi i \nu_n \varphi} e^{-\nu_n s} \quad \text{and} \quad \sum_{n=0}^{\infty} b_{\nu_n} G(\mu_n) e^{2\pi i \nu_n \varphi} e^{-\nu_n s}
\]
must have the same convergence abscissa as (4.1). For definiteness suppose the first to be true. Then, as easily to be seen from the Kojima's theorem,\(^{1)}\)
\[
\sum_{n=0}^{\infty} R \left( - i c_{\nu_n} G(\mu_n) e^{2\pi i \psi_n} \right) e^{-\nu_n s}
\]
has the same convergence abscissa as (4.3). By Theorem 1 \( s = a \) is a singular point of (4.2). That is, the points
\[
s = a + (p' \varphi + 2n\pi)i \quad (p' \equiv p' \mod 2\pi; \quad p, n = 1, 2, \ldots)
\]
are singular points of (4.1) and then of (1.2). Since the point set (4.4) is everywhere dense on the convergence line \( \sigma = a \), this line is the singular line. So we have

**Theorem 2.** Suppose that the Dirichlet's series (1.2) has the finite convergence abscissa \( \sigma = a \), and
\[
0 \leq \arg c_{\lambda_n} \leq \frac{\pi}{2}
\]
with the exception of \( c_{\lambda_n} \) such that
\[
\frac{r_{\nu}}{\nu} \to \infty, \quad \lim_{\nu \to \infty} (r_{\nu} - r_{\nu-1}), \quad \lim_{\nu, \kappa \to \infty} (r_{\nu} - \lambda_{\kappa}) > 0,
\]
suppose further that \( \lim_{\nu \to \infty} e^{2\pi i \mu_n \varphi} \) exists for some irrational number \( \varphi \) and for the complementary set \( (\mu_n) \) of \( (\nu_n) \). Then the series (1.2) has the convergence line as the singular line.

This is a generalization of Gergen-Widder's theorem.

\(^{1)}\) Gergen-Widder, Am. Journ. of Math. 50 (1928).