93. On the Forced Vibration of an Elastic Rod.

By Seiichi HIGUCHI.

Tohoku Imperial University.

(Comm. by Kyoji SUEHIRO, M.I.A., Oct. 13, 1930.)

§ 1. The problem of the flexural vibration of an elastic rod, in both the cases when the internal resistance of the rod is taken into account and when it is not, has been discussed in many ways by many authors.¹⁾

In the present paper, at the suggestion of Prof. Iwao Kobayashi, the writer deals with the problem of the forced vibration of an elastic rod intending to establish a solution from which we can determine the state of the rod, easily and clearly, not only its initial state but also the subsequent configuration.

The general case which is treated here is as follows:—"The flexural vibration of an elastic rod whose length is l , subjected to internal resistance clamped at one end, while a body whose mass is M is attached to the free end, under the following external force which acts at the clamped end,

$$i\bar{A}e^{ipt}$$

where $\bar{A}$ is real and $I_m p > 0$, provided that the rod initially has neither displacement nor velocity."

§ 2. The equation of motion, boundary and initial conditions of the case are as follows:

$$\begin{aligned} (1) \quad & \frac{\partial^2 y}{\partial t^2} + \frac{EK^2}{\rho} \left\{ \frac{\partial^4}{\partial x^4} + \frac{\mu}{E} \frac{\partial^5}{\partial t \partial x^4} \right\} y = 0^{2)} & 0 \leq x \leq l. \\ (2) \quad & \left. \begin{aligned} y &= i\bar{A}e^{ipt} & t > 0 \\ y &= 0 & t < 0 \end{aligned} \right\} & x = 0, \\ (3) \quad & \dot{y}_x = 0 & x = 0, \\ (4) \quad & -E\omega k^2 \left\{ \frac{\partial^2}{\partial x^2} + \frac{\mu}{E} \frac{\partial^3}{\partial t \partial x^2} \right\} y = \oplus \frac{\partial^2}{\partial t^2} \left(\frac{\partial y}{\partial x} \right) & x = l, \\ (5) \quad & E\omega k^2 \left\{ \frac{\partial^3}{\partial x^3} + \frac{\mu}{E} \frac{\partial^4}{\partial t \partial x^3} \right\} y = M \frac{\partial^2 y}{\partial t^2} & x = l, \\ (6) \quad & y = 0 & 0 \leq x \leq l, \\ (7) \quad & \left. \begin{aligned} \dot{y}_t &= 0 & \left\{ \begin{aligned} t &\leq 0 \\ t &< 0 \end{aligned} \right\} \end{aligned} \right\} & \begin{aligned} 0 < x \leq l, \\ 0 = x \end{aligned} \end{aligned}$$

1) For example, H. Horler: Z. f. ang. Mathem. u. Mech. Bd. 8 (1928). K. Suyehiro: Proc. 4 (1928). S. Miyazawa: Sci. Rep. Tohoku Imp. Univ. Vol. 19 (1930). K. Muto: Z. f. ang. Mathem. u. Mech. Bd. 10 (1930).

2) The axis of the rod in the equilibrium position is taken as x and a axis perpendicular to it as y . And in this paper, the effects of air resistance, the rotary inertia and the longitudinal tension of the rod etc., are assumed to be negligible.

where E , ρ , ω , μ , k and $\mathbb{H}$ denote respectively Young's modulus, density, area of cross section, coefficient of internal viscosity, radius of gyration of the cross section about an axis through its centroid and moment of inertia of mass M about an axis through the end of the rod, both the axes being at right angles to the plane of vibration. And as already mentioned, p is taken as $I_m p > 0$.

Put

$$(8) \quad y = \frac{i\bar{A}}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{i\alpha t} d\alpha}{(\alpha - p)F(\alpha)} [A \cos \xi(l-x) + B \sin \xi(l-x) + C \operatorname{Cos} \xi(l-x) + D \operatorname{Sin} \xi(l-x)]$$

where

$$(8_1) \quad \begin{cases} A = -PQ(\cos \xi l - \operatorname{Cos} \xi l) + 2P \operatorname{Sin} \xi l + (\cos \xi l + \operatorname{Cos} \xi l) \\ B = -PQ(\sin \xi l + \operatorname{Sin} \xi l) - 2Q \operatorname{Cos} \xi l + (\sin \xi l - \operatorname{Sin} \xi l) \\ C = PQ(\cos \xi l - \operatorname{Cos} \xi l) + 2P \sin \xi l + (\cos \xi l + \operatorname{Cos} \xi l) \\ D = PQ(\sin \xi l + \operatorname{Sin} \xi l) + 2Q \cos \xi l + (\sin \xi l - \operatorname{Sin} \xi l) \end{cases}$$

$$(8_2) \quad F(\alpha) = 2[1 + \cos \xi l \operatorname{Cos} \xi l + Q(\operatorname{Sin} \xi l \cos \xi l - \operatorname{Cos} \xi l \sin \xi l) - PQ(1 - \cos \xi l \operatorname{Cos} \xi l) + P(\operatorname{Sin} \xi l \cos \xi l + \operatorname{Cos} \xi l \sin \xi l)]$$

$$(8_3) \quad \xi^4 = \alpha^2 \left/ \frac{Ek^2}{\rho} \right(1 + \frac{\mu}{E} \alpha)$$

$$(8_4) \quad \begin{cases} Q = M\xi l / \rho \omega l \\ P = -\mathbb{H} \xi^3 / \rho \omega \end{cases} \quad \text{and} \quad (8_5) \quad \operatorname{Cos} \xi l = \cos i\xi l, \quad \operatorname{Sin} \xi l = -i \sin i\xi l.$$

Here, it is easily verified that all the roots of $F(\alpha) = 0$ are either complex and $I_m \alpha > 0$ or purely imaginary,¹⁾ which is the same as to say, the poles of the integrand of (8) lie fully in the upper half circle as is expressed in Fig. 1.

Introducing the path of Fig. 1, we have in the place of (8),

$$(9) \quad y = \frac{i\bar{A}}{2\pi i} \int \frac{e^{i\alpha t} d\alpha}{(\alpha - p)F(\alpha)} [A \cos \xi(l-x) + B \sin \xi(l-x) + C \operatorname{Cos} \xi(l-x) + D \operatorname{Sin} \xi(l-x)]$$

over the path of Fig. 1.

The proof that (8) or (9) satisfies all necessary boundary and initial conditions is here omitted.

Thus, the solutions given in (9) reduces, by using the path of Fig. 1 and Theorem of residue, to

$$(10) \quad y = i\bar{A} \sum \frac{e^{i\alpha t}}{\alpha - p} \frac{1}{F'(\alpha)} [A \cos \xi(l-x) + B \sin \xi(l-x) + C \operatorname{Cos} \xi(l-x) + D \operatorname{Sin} \xi(l-x)]$$

1) Here, it is first proved that the roots ξ of $F(\alpha) = 0$ are either real or purely imaginary.

$$+ i\bar{A} | e^{i\alpha t}/F(\alpha)[A \cos \xi(l-x) + B \sin \xi(l-x) + C \operatorname{Cos} \xi(l-x) \\ + D \operatorname{Sin} \xi(l-x)] |_{\alpha=p}$$

where summation is taken over all the roots of $F(\alpha)=0$. The convenient form for numerical calculation is easily derived from (10), but it is not written here.

§ 3. In this section, we are to treat the following special cases:—
(A). The case when the body at the free end of the rod is considered to be a material point whose mass is M .

In this case, the equation of motion, boundary and initial conditions are the same as are obtained in § 2, except that (4) must be written as

$$\ddot{y}_x=0 \quad \text{at } x=l.$$

Now, using the method similar to that of § 2, we obtain the solution

$$(11) \quad y = \frac{i\bar{A}}{2\pi i} \int \frac{e^{i\alpha t} d\alpha}{(\alpha-p)F(\alpha)} [(\cos \xi l + \operatorname{Cos} \xi l) \{ \cos \xi(l-x) + \operatorname{Cos} \xi(l-x) \} \\ + (\sin \xi l - \operatorname{Sin} \xi l + Q \operatorname{Cos} \xi l) \sin \xi(l-x) + (\sin \xi l - \operatorname{Sin} \xi l \\ - Q \cos \xi l) \operatorname{Sin} \xi(l-x)]$$

where $Q = M\xi l/\rho\omega l$ and

$$F(\alpha) = 2[1 + \cos \xi l \operatorname{Cos} \xi l + Q(\cos \xi l \operatorname{Sin} \xi l - \sin \xi l \operatorname{Cos} \xi l)]$$

The solution given in (11) also reduces, by using the path of Fig. 1 and Theorem of residue, to

$$(12) \quad y = i\bar{A} \sum \frac{e^{i\alpha t}}{\alpha-p} \frac{1}{F'(\alpha)} [(\cos \xi l + \operatorname{Cos} \xi l) \{ \cos \xi(l-x) + \operatorname{Cos} \xi(l-x) \} \\ + (\sin \xi l - \operatorname{Sin} \xi l + Q \operatorname{Cos} \xi l) \sin \xi(l-x) + (\sin \xi l - \operatorname{Sin} \xi l \\ - Q \cos \xi l) \operatorname{Sin} \xi(l-x)] \\ + i\bar{A} | e^{i\alpha t}/F(\alpha)[(\cos \xi l + \operatorname{Cos} \xi l) \{ \cos \xi(l-x) + \operatorname{Cos} \xi(l-x) \} \\ + (\sin \xi l - \operatorname{Sin} \xi l + Q \operatorname{Cos} \xi l) \sin \xi(l-x) + (\sin \xi l - \operatorname{Sin} \xi l \\ - Q \cos \xi l) \operatorname{Sin} \xi(l-x)] |_{\alpha=p}$$

summation being taken over all the roots of $F(\alpha)=0$.

(B). The case when $M=0$ and $I_m p=0$.

In this case, we have to revise the boundary conditions which are obtained in § 2 as is shown below:

$$\left. \begin{aligned} y &= i\bar{A}e^{ipt} & t > 0 \\ y &= 0 & t < 0 \end{aligned} \right\} x=0,$$

where p =real positive, and

$$\ddot{y}_x=0 \quad x=l.$$

Introducing the path of Fig. 2, we have the solution

$$(13) \quad y = \frac{i\bar{A}}{2\pi i} \int \frac{e^{i\alpha t} d\alpha}{(\alpha-p)F(\alpha)} [(\cos \xi l + \operatorname{Cos} \xi l) \{ \cos \xi(l-x) + \operatorname{Cos} \xi(l-x) \} \\ + (\sin \xi l - \operatorname{Sin} \xi l) \{ \sin \xi(l-x) + \operatorname{Sin} \xi(l-x) \}]$$

where $F(a) = 2(1 + \cos \xi l \cos \xi l)$.

And finally,

$$(14) \quad y = i\bar{A} \sum \frac{e^{i\alpha t}}{a} \frac{1}{p} \frac{1}{F'(a)} [(\cos \xi l + \cos \xi l) \{ \cos \xi(l-x) + \cos \xi(l-x) \} \\ + (\sin \xi l - \sin \xi l) \{ \sin \xi(l-x) + \sin \xi(l-x) \}] \\ + i\bar{A} \left| \frac{e^{i\alpha t}}{F(a)} [(\cos \xi l + \cos \xi l) \{ \cos \xi(l-x) + \cos \xi(l-x) \} \right. \\ \left. + (\sin \xi l - \sin \xi l) \{ \sin \xi(l-x) + \sin \xi(l-x) \}] \right|_{\alpha=p}$$

where, also in this case, the summation is taken over all the roots of $F(a) = 0$.

The form from which we can calculate the numerical values of y is easily derived from (12) and (14), and the process of this treatment is far simpler than that of the usual one.

§ 4. The values of ξl and at the same time that of α , involve many important and useful significances. But here we will not enter fully into the discussion of it, except one example, relating directly to our mathematical treatment.

Since the roots ξ of $F(a) = 0$ in (8), (11) and (13) are either real or purely imaginary, we see that ξ^4 must be real and positive.

Hence from the expression of ξ^4 we have

$$\alpha = \frac{1}{2} [ib\xi^4 \pm \xi^2 \sqrt{-b^2\xi^4 + 4a}]$$

where $a = Ek^2/\rho$ and $b = \mu k^2/\rho$.

Then we have the following two cases i.e.,¹⁾

case 1. $b^2\xi^4 < 4a$ then $\alpha = \text{complex}$ and $I_m\alpha > 0$ (damped oscillation)

case 2. $b^2\xi^4 \geq 4a$ $\alpha = \text{purely imaginary}$
 $= i\alpha'$ and $\alpha' > 0$ (no oscillation).

In the above, the first case plainly corresponds to the case of our treatment in which the path of integration of (9), (11) and (13) is expressed in Fig. 1 and Fig. 2, but the poles corresponding to the later case distribute as is shown in Fig. 3.²⁾

1) A physical interpretation of the case corresponding to that of (B) in § 3 has been already published by Prof. Suyehiro (loc. cit.).

2) On the problem cited here, a somewhat detailed discussion has been published by A. Kneschke. (Ann. d. Phys. 2 (1929), S. 555.)