46. Repeated Stress in Railway Rails.

By Akimasa ONO.

Institute of Strength of Materials, Kyushu Imperial University, Fukuoka.

(Comm. by K. TAWARA, M.I.A., May 12, 1932.)

The transverse fissure, which occurs in railway rails, shows generally the characteristic feature of fatigue fracture, and the position of the fissure in a rail is usually found to be near an end.¹⁾ As the position is localized, due attention should be paid to the effect of the rail-joint, which was already made the subject of investigation in another paper,²⁾ from which equations may be taken so far as they are necessary for the present purpose. If we consider a rail as a bar placed on a continuous foundation, the deflection of the elastic line may be represented by the following equations, the origin of the axes x and y being taken at the joint.

$$\left. \begin{aligned} x < 0, & \quad y = e^{\beta x}(A_1 \cos \beta x + B_1 \sin \beta x), \\ \lambda > x > 0, & \quad y = e^{\beta x}(A_2 \cos \beta x + B_2 \sin \beta x) + e^{-\beta x}(C_2 \cos \beta x + D_2 \sin \beta x), \\ \xi = x - \lambda > 0, & \quad y = e^{-\beta \xi}(C_3 \cos \beta \xi + D_3 \sin \beta \xi), \quad \text{where } \beta = \sqrt[4]{\frac{K}{4EJ_1}}, \end{aligned} \right\} \quad (1)$$

K being the modulus of foundation and EJ_1 the flexural rigidity of the rail.

The constants in (1) run as follows.

$$\left. \begin{aligned} A_1 &= 2P'e^{-\beta \lambda} \left\{ \left(\frac{2+q}{4+q} - \frac{p}{2+p} \right) \cos \beta \lambda + \frac{2}{4+q} \sin \beta \lambda \right\}, \\ B_1 &= -P'e^{-\beta \lambda} \frac{4}{4+q} (\cos \beta \lambda - \sin \beta \lambda), \\ A_2 &= P'e^{-\beta \lambda} (\cos \beta \lambda + \sin \beta \lambda), \\ B_2 &= -P'e^{-\beta \lambda} (\cos \beta \lambda - \sin \beta \lambda), \\ C_2 &= P'e^{-\beta \lambda} \left\{ \left(\frac{2p}{2+p} + \frac{q}{4+q} \right) \cos \beta \lambda - \frac{q}{4+q} \sin \beta \lambda \right\}, \\ D_2 &= -P'e^{-\beta \lambda} \frac{q}{4+q} (\cos \beta \lambda - \sin \beta \lambda), \\ C_3 &= P' \left\{ 1 + e^{-2\beta \lambda} \left[\frac{p}{2+p} (1 + \cos 2\beta \lambda) + \frac{q}{4+q} (1 - \sin 2\beta \lambda) \right] \right\}, \end{aligned} \right\} \quad (2)$$

1) According to a report by Mr. Kohira, research staff of the Steel Works, Yawata.

2) Zeitschrift f. Angewandte Mathematik u. Mechanik, Bd. 11, 1931, p. 165.

$$D_3 = P' \left\{ 1 - e^{-2\beta\lambda} \left[\frac{q}{4+q} \cos 2\beta\lambda + \frac{p}{2+p} \sin 2\beta\lambda \right] \right\}. \quad \Bigg\}$$

In these expressions

$$P' = \frac{P}{8\beta^3 EJ_1}, \quad p = \frac{2\beta^3 J_1}{a^3 J_2}, \quad q = \frac{4\beta J_1}{a J_2}, \quad \text{with } a = \sqrt[4]{\frac{K_1}{4E} \left(\frac{1}{J_1} + \frac{1}{J_2} \right)}, \quad (3)$$

where K_1 is the modulus of joint denoting the intensity of pressure acting between the rail and fish-plates beyond the initial stress in the condition in which the rail and the fish-plates are relatively displaced by unit distance at the place under consideration.

The bending moment acting at a section in the region $x > \lambda$ is given by the last equation of (1), viz.,

$$M = -EJ_1 \frac{d^2 y}{dx^2} = -2\beta^2 EJ_1 e^{-\beta x} (C_3 \sin \beta x - D_3 \cos \beta x). \quad (4)$$

Differentiating this expression with respect to x , we have for $x = \lambda$

$$\begin{aligned} \left. \frac{dM}{dx} \right|_{\lambda} &= -2\beta^3 EJ_1 (C_3 + D_3) \\ &= -\frac{P}{2} \left\{ 1 + \frac{1}{2} e^{-2\beta\lambda} \left[\frac{p}{2+p} + \frac{q}{4+q} \right. \right. \\ &\quad \left. \left. - \left(\frac{q}{4+q} - \frac{p}{2+p} \right) \cos 2\beta\lambda - \left(\frac{p}{2+p} + \frac{q}{4+q} \right) \sin 2\beta\lambda \right] \right\}. \end{aligned}$$

With the numerical values of p , q and β (loc. cit.), it will be seen that this expression is negative. Accordingly, the bending moment decreases with x in the vicinity of the load.

Again we have for $\lambda > x > 0$

$$\begin{aligned} \left. \frac{dM}{dx} \right|_{\lambda} &= \frac{P}{2} \left\{ 1 - \frac{1}{2} e^{-2\beta\lambda} \left[\frac{p}{2+p} + \frac{q}{4+q} \right. \right. \\ &\quad \left. \left. - \left(\frac{q}{4+q} - \frac{p}{2+p} \right) \cos 2\beta\lambda - \left(\frac{p}{2+p} + \frac{q}{4+q} \right) \sin 2\beta\lambda \right] \right\}. \end{aligned}$$

This expression being positive, the bending moment increases with x . Thus the moment is greatest at the point $x = \lambda$, and its value is given by (2) and (4), viz.,

$$M_1 = 2\beta^2 EJ_1 D_3 = \frac{P}{4\beta} \left\{ 1 - e^{-2\beta\lambda} \left[\frac{q}{4+q} \cos 2\beta\lambda + \frac{p}{2+p} \sin 2\beta\lambda \right] \right\}, \quad (5)$$

x being written instead of λ .

Next the bending moment in the region $x < 0$ is from (1) and (2)

$$\begin{aligned} M = -EJ_1 \frac{d^2 y}{dx^2} &= -\frac{P}{2\beta} e^{\beta(x-\lambda)} \left\{ \left(\frac{p}{2+p} - \frac{q}{4+q} \right) \cos \beta\lambda \sin \beta x \right. \\ &\quad \left. + \frac{2}{4+q} \left[\sin \beta(\lambda-x) - \cos \beta(\lambda-x) \right] \right\}. \end{aligned}$$

In order to find the position of the load corresponding to a maximum

of the magnitude of the moment, we have $\frac{\partial M}{\partial \lambda} = 0$, from which

$$\tan \beta \lambda = -\frac{1}{4+q} \left[2(2+p) \cot \beta x + q - 2p \right].$$

Several sets of numerical values of λ and x satisfying this condition for given values of p , q and β being calculated, it is found that the two quantities are approximately in a linear relation in a certain range of x , i.e., $\beta \lambda = \beta x + \varepsilon$ ($x < 0$). Substituting this relation in the expression for M , we find the required moment, which refers to a point on the negative side of x , the load being on the positive side. For a point on the positive side, the moment due to a load on the negative side is

$$M_2 = -\frac{P}{4\beta} e^{-\varepsilon} \left\{ \left(\frac{q}{4+q} - \frac{p}{2+p} \right) [\sin (2\beta x - \varepsilon) + \sin \varepsilon] + \frac{4}{4+q} (\sin \varepsilon - \cos \varepsilon) \right\}. \quad (6)$$

In the case when the joint is quite loose, $\alpha = 0$ and $p = q = \infty$. Then

$$M_1 = \frac{P}{4\beta} \left\{ 1 - e^{-2\beta x} [\cos 2\beta x + \sin 2\beta x] \right\}, \quad (7)$$

and the value of (4) in the case, when $p = q = \infty$ and $\lambda = 0$, is

$$M_2 = -\frac{P}{\beta} e^{-\beta x} \sin \beta x. \quad (8)$$

Now the stresses caused at the top layer of a rail are $\sigma_1 = -\frac{M_1 e}{J_1}$

and $\sigma_2 = -\frac{M_2 e}{J_1}$, e being the distance of the top layer from the neutral axis.

The mean and the half range of stresses are in magnitudes

$$\frac{1}{2} |\sigma_1 + \sigma_2| = \frac{1}{2} \frac{e}{J_1} |M_1 + M_2| \quad \text{and} \quad \frac{1}{2} |\sigma_1 - \sigma_2| = \frac{1}{2} \frac{e}{J_1} (M_1 - M_2),$$

respectively. These quantities may be taken as factors determining the number of stress cycles repeated before fracture. If we take a linear relation between these two as a law for the fatigue limit, then we have

$$\sigma_a = \frac{1}{2} |\sigma_1 - \sigma_2| + \frac{c}{2} |\sigma_1 + \sigma_2|, \quad \text{where } \sigma_a \text{ is an equivalent stress alternating}$$

between a positive and a negative limit of equal magnitudes, and

c is a constant. Putting $\sigma_a = M_a \frac{e}{J_1}$, we have

$$M_a = \frac{1}{2} (M_1 - M_2) \pm \frac{c}{2} (M_1 + M_2). \quad (9)$$

The upper or the lower sign before the constant c should be taken according as $(M_1 + M_2)$ is positive or negative.

Lastly, to find the value of x , which makes M_a a maximum, we have

$$(1 \pm c) \frac{dM_1}{dx} = (1 \mp c) \frac{dM_2}{dx}.$$

Accordingly, from (5) and (6)

$$\begin{aligned} \frac{1+c}{1-c} e^{\varepsilon-2\beta x} \left\{ \frac{q}{4+q} (\cos 2\beta x + \sin 2\beta x) + \frac{p}{2+p} (\sin 2\beta x - \cos 2\beta x) \right\} \\ + \left(\frac{q}{4+q} - \frac{p}{2+p} \right) \cos (2\beta x - \varepsilon) = 0, \end{aligned} \quad (10)$$

as $M_1 + M_2$ is obviously positive, and in the case when $p = q = \infty$, from (7) and (8)

$$\frac{1-c}{1+c} e^{-\beta x} = \frac{1}{2} \left(\frac{1}{\cos \beta x} - \frac{1}{\sin \beta x} \right), \quad (11)$$

as $M_1 + M_2$ is negative in the part of x under consideration.

The value of the constant c depends on the quality of material. For example, nine series of endurance tests on annealed steels of various carbon contents (0.10 to 0.72 p. cent.) gave the average result that c is equal to 0.86 for the positive value of the mean stress. The same value of c was taken in the calculation both for the positive and the negative mean stress. The result of the calculation runs as follows.

Section of rails	37 k	50 k PS
ε in (10).....	1.327	1.373
x Distance of breaking section, cm		
If joint is very tight	113.7	132.2
If joint is quite loose	75.6	87.9
M_a Equivalent bending moment (9)		
If joint is very tight	$1.918 \frac{P}{8\beta}$	$1.913 \frac{P}{8\beta}$
If joint is quite loose	$2.511 \frac{P}{8\beta}$	$2.511 \frac{P}{8\beta}$

This result may be compared with the case of rails without joint. In this case we have $M_1 = \frac{P}{4\beta}$ and $M_2 = -\frac{P}{4\beta} e^{-\frac{\pi}{2}}$. These values of moments give $M_a = 1.889 \frac{P}{8\beta}$. It is to be observed that the equivalent moment in rails with a tight joint is only slightly greater than that in rails without joint, but the moment in rails with a joint quite slackened is some 33 p. cent. greater than the latter. Moreover, the values of x for the breaking section as given in the above table may explain the localization of the fissure.