12. On the Cartan Decomposition of a Lie Algebra.

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Let $\mathfrak{L}$ be a Lie algebra over the field of complex numbers, $\mathfrak{H}$ and $\mathfrak{H}'$ maximal nilpotent subalgebras of $\mathfrak{L}$ containing regular elements. E. Cartan has shown for semi-simple $\mathfrak{L}$ that there exists an inner automorphism A such that $\mathfrak{H}' = A\mathfrak{H}$ ¹⁾. In this note we shall show that this theorem is valid for any, not necessarily semi-simple, Lie algebra. From this we see easily that the decomposition of a Lie algebra into the eigen-spaces of a maximal nilpotent subalgebra containing a regular element (Cartan decomposition) is unique up to inner automorphisms of $\mathfrak{L}$.

Let $\mathfrak{G}$ be the Lie group which corresponds to $\mathfrak{L}$. To every element a of $\mathfrak{L}$ corresponds a one-parameter subgroup $g(t)$ of $\mathfrak{G}$ and a is the tangent vector at the unit element to the differentiable curve $g(t)$. Extending to general Lie groups a notion familiar for matrices, we shall denote by $\exp ta$ this one-parameter subgroup $g(t)$ and by $\exp a$ the point of parameter 1 on this curve. Further $\exp \mathfrak{H}$ means the (local) subgroup of $\mathfrak{G}$ which corresponds to a subalgebra $\mathfrak{H}$ of $\mathfrak{L}$. If we transform the elements of the group $\exp ta$ by a fixed element g , we obtain a new one-parameter subgroup $\exp ta'$; the mapping $a \rightarrow A_g a = a'$ is an inner automorphism of $\mathfrak{L}$ generated by g . The mapping $x \rightarrow D_a x = [a, x]$, with a fixed, is a linear operation in $\mathfrak{L}$, which is called inner derivation of $\mathfrak{L}$. Suppose that $g = \exp a$ and g is sufficiently near to the unit element, then $A_g = \exp D_a$. Let us decompose $\mathfrak{L}$ by A_g into eigen-spaces:

$$\mathfrak{L} = \mathfrak{L}_1 + \mathfrak{L}_\rho + \mathfrak{L}_\sigma + \dots$$

where $\mathfrak{L}_1, \mathfrak{L}_\rho, \dots$ are the eigen-spaces for the characteristic roots, 1, $\rho, \dots$ of A_g . Here $\mathfrak{L}_1$ is a subalgebra of $\mathfrak{L}$ ²⁾.

Lemma³⁾. The systems $u^{-1}g \exp \mathfrak{L}_1 u$, where u runs over a neighbourhood of the unit element, contain a neighbourhood of the element g in $\mathfrak{G}$.

Proof. Let $a_1, a_2, \dots, a_s$ be a basis of the subalgebra $\mathfrak{L}_1$ and $a_{s+1}, \dots, a_r$ a basis of $\mathfrak{L}_\rho + \mathfrak{L}_\sigma + \dots$. Then the (local) subgroup $\exp \mathfrak{L}_1$ is composed of all elements of the forms $\exp(t_1 a_1 + \dots + t_s a_s)$, where the parameters t_i are sufficiently near to zero. To prove our Lemma, it is sufficient to show that the set of elements

1) E. Cartan, Le principe de dualité et la théorie des groupes simples et semi-simples (Bull. Sc. math. t. 49, 1925). Gantmacher has given a proof in a somewhat general form. F. Gantmacher, Canonical representations of automorphisms of a complex semi-simple Lie group, (Recueil mathématique, 5(47), 1939).

2) See Gantmacher, l. c. P. 107: If g is sufficiently near to the unit element then $\mathfrak{L}_1 \neq 0$.

3) Cf. Gantmacher, l. c. P. 113.

$$g^{-1} \exp(-(t_{s+1}a_{s+1} + \dots + t_r a_r)) g \exp(t_1 a_1 + \dots + t_s a_s) \exp(t_{s+1}a_{s+1} + \dots + t_r a_r) \\ = \exp(p_1 a_1 + p_2 a_2 + \dots + p_r a_r),$$

where $p_i = p_i(t_1, \dots, t_r)$ are analytic functions of t_k , cover a neighbourhood of the unit element, when t_k run independently over a neighbourhood of zero. To see this it suffices to show that the Jacobian

$$\frac{\partial(p_1 \dots p_r)}{\partial(t_1 \dots t_r)}$$

is different from zero for $t_1 = t_2 = \dots = t_r = 0$.

Let $t_i = 0$ for $i \neq j$, $1 \leq j \leq s$. Then

$$\exp(t_j a_j) = \exp \sum_{i=1}^r p_i(0 \dots t_j \dots 0) a_i.$$

Hence $p_i(0 \dots t_j \dots 0) = \delta_{ij} t_j$ and $\left(\frac{\partial p_i}{\partial t_j}\right)_{t=0} = \delta_{ij}$, for $1 \leq j \leq s$.

Now, let $t_i = 0$ for $i \neq j$, $j > s$. Then

$$g^{-1} \exp(-t_j a_j) g \cdot \exp(t_j a_j) = \exp(-t_j A_g a_j) \exp(t_j a_j) \\ = \exp\left(\sum_{i=1}^r p_i(0 \dots t_j \dots 0) a_i\right).$$

From this we obtain the equations

$$(1 - A_g) a_j = \sum_{i=1}^r \left(\frac{\partial p_i}{\partial t_j}\right)_{t=0} a_i, \text{ for } j > s.$$

Since the linear operation $1 - A_g$ transforms the space $\mathfrak{L}_p + \mathfrak{L}_s + \dots$ into itself and is non singular on this space, $\left(\frac{\partial p_i}{\partial t_j}\right)_{t=0} = 0$ for $1 \leq i \leq s$, $j > s$, and the matrix

$$\left(\left(\frac{\partial p_i}{\partial t_j}\right)_{t=0}\right)_{s+1 \leq i, j \leq r}$$

is non-singular.

Thus $\frac{\partial(p_1 \dots p_r)}{\partial(t_1 \dots t_r)} \neq 0$.

Now let $a_1, a_2, \dots, a_r$ be a basis of $\mathfrak{L}$, $\xi_1 a_1 + \xi_2 a_2 + \dots + \xi_r a_r$ a general element of $\mathfrak{L}$ ($\xi_1, \dots, \xi_r$ are variables) and let

$$|tE - (\xi_1 D_{a_1} + \dots + \xi_r D_{a_r})| = t^r - \psi_1(\xi) t^{r-1} + \dots \pm \psi_{r-1}(\xi) t$$

be the characteristic equation of $\mathfrak{L}$.

An element $a = \sum_{i=1}^r \lambda_i a_i$ of $\mathfrak{L}$ is called regular, if $\psi_{r-1}(\lambda) \neq 0$.

The totality of regular elements is an open set in r dimensional complex vector space $\mathfrak{L}$ and singular elements form an algebraic manifold of at most $r-1$ complex dimensions. Hence the set of all regular elements is connected.

Let $a = \sum_{i=1}^r \lambda_i a_i$ and $b = \sum_{i=1}^r \mu_i a_i$ be two regular elements, $\mathfrak{H}_a$ and $\mathfrak{H}_b$ the maximal nilpotent subalgebras of $\mathfrak{L}$ containing a and b respective-

ly. First let the parameters (μ_i) be sufficiently near to (λ_i) . We choose a sufficiently small positive number ξ such that $D_{\xi a} = \xi D_a$ has no characteristic roots of the form $2\pi\sqrt{-1}m$, where m denotes integer $\neq 0$. Let $g = \exp(\xi a)$ and $\mathfrak{L}_1$ be the eigen-space for the characteristic root 1 of the inner automorphism A_g . Since $\mathfrak{H}_a$ is the eigen-space for the characteristic root 0 of the inner derivation $D_{\xi a}$, $A_g = \exp D_{\xi a}$ and, moreover, $D_{\xi a}$ has no characteristic root of the form $2\pi\sqrt{-1}m$, we have $\mathfrak{L}_1 = \mathfrak{H}_a$. As $\exp \xi b$ is sufficiently near to g , there exists, by the above lemma, an element $u \in \mathfrak{G}$ such that $\exp \xi b \exp u^{-1} g \exp \xi a u$. But since g is contained in $\exp \mathfrak{H}_a$, we have $g \exp \mathfrak{H}_a \leq \exp \mathfrak{H}_a$. Hence $\exp \xi b \exp u^{-1} \exp \mathfrak{H}_a u = \exp A_u \mathfrak{H}_a$. Thus there exists an element $c \in \mathfrak{H}_a$, which is also regular such that $\xi b = A_u c$. Then we obtain

$$\mathfrak{H}_b = \mathfrak{H}_{\xi b} = \mathfrak{H}_{A_u c} = A_u \mathfrak{H}_c,$$

and since $c \in \mathfrak{H}_a$, $\mathfrak{H}_c = \mathfrak{H}_a$. Thus $\mathfrak{H}_b = A_u \mathfrak{H}_a$. Now let a and b be arbitrary regular element. Since the set of all regular elements is connected, we see by continuity that there exists an inner automorphism A such that $A \mathfrak{H}_a = \mathfrak{H}_b$.

Thus we have proved the following.

Theorem. Let $\mathfrak{H}$ and $\mathfrak{H}'$ be two maximal nilpotent subalgebras containing regular elements of a Lie algebra $\mathfrak{L}$ over the field of complex numbers. Then there exists an inner automorphism A such that $\mathfrak{H}' = A \mathfrak{H}$. The Cartan decomposition of $\mathfrak{L}$ is unique up to inner automorphisms of $\mathfrak{L}$.

Added in proof (May 2, 1950).

After the present note was submitted to the Proc. of Acad. of Japan, I was made aware through Mathematical Reviews that the result in the present note had been already proved by C. Chevalley in his paper, "An algebraic proof of a property of Lie groups," Amer. J. Math. v. 63 (1941). But I assume that my approach is different from his.