Suppose that a pair of pitch curves $K_1$ and $K_2$ and a pair of profile curves $F_1$ and $F_2$ invariably connected with $K_1$ and $K_2$ be given. Let $P$ be a common pitch point at a certain instant, $C$ be the point of contact of $F_1$ and $F_2$ corresponding to $P$. Suppose that after infinitesimal time interval $dt$ two points $P_1$ and $P_2$ on $K_1$ and $K_2$ and two points $C_1$ and $C_2$ on $F_1$ and $F_2$ may respectively come to the point of contact. Denote by $ds$ the length of the arc $PP_1$, and consequently that of $PP_2$, and by $dp_1$ and $dp_2$ the lengths of the arcs $CC_1$ and $CC_2$ respectively. The pitch curve $K$ is oriented and accordingly $ds$ has a sign positive or negative. We shall give also a sign to $dp$; $dp$ is positive or negative according as the part of arc $dp$ of the profile curve $F$ is of positive or negative type.

§ 1. Sliding of profile curves.

At the sliding contact motion of $F_1$ and $F_2$ during the time $dt$ the point $C$ on $F_1$ slides along $F_2$ for the distance $dp_2 - dp_1$, and consequently its velocity $v_{p1}$ is given by

$$(1) \ v_{p1} = \frac{dp_2 - dp_1}{dt}. \quad (1)_1$$

$v_{p1}$ is named the velocity of sliding of $F_1$ (at the point $C$ on $F_2$). In like manner the velocity of sliding of $F_2$ may be defined:

$$(1) \ v_{p2} = \frac{dp_1 - dp_2}{dt}. \quad (1)_2$$

Evidently $v_{p1}$ and $v_{p2}$ have the same absolute value and the different signs.

Denoted by $\omega_1$ and $\omega_2$ respectively the instant angular velocities of $K_1$ and $K_2$ at their rolling contact motions, and we say that the sign of the angular velocity $\omega$ is positive or negative according as $K$ rotates clockwise or counterclockwise. Denoting by $a_1$ and $a_2$ the radii of curvature of $K_1$ and $K_2$ respectively at the instant common pitch point $P$ we have

$$(2) \ \omega_1 = \frac{1}{a_1} \frac{ds}{dt}, \quad \omega_2 = \frac{1}{a_2} \frac{ds}{dt}. \quad (2)$$

Let $\omega$ denote the relative rolling angular velocity of $K_1$ to $K_2$, then obviously $\omega = \omega_1 - \omega_2$ and accordingly from (2)
Next, let \( r \) be the length of the segment of the straight line connecting \( P \) with the point of contact \( C \) of \( F_1 \) and \( F_2 \), and give \( r \) a positive or negative sign in such a manner as we have explained at the beginning of the report (II), then the velocity \( v_{pl} \) of \( C \) is represented by \( r \omega \), that is,

\[
(4) \quad v_{pl} = \left( \frac{1}{a_1} - \frac{1}{a_2} \right) r \frac{ds}{dt}.
\]

Similarly

\[
(4) \quad v_{p2} = \left( \frac{1}{a_2} - \frac{1}{a_1} \right) r \frac{ds}{dt}.
\]

From (4) follows immediately:

Profile curves make rolling contact motion without sliding if and only if they coincide with their pitch curves.

Now, we may consider the acceleration of \( C \): \( 
\omega_p \). Denoting by \( w_t \) and \( w_n \) its tangential and normal component, we have, as is well known,

\[
(5) \quad w_t = \frac{dv_p}{dt}, \quad w_n = \frac{v_p^2}{m},
\]

where \( m \) denotes the radius of curvature of the mate of \( F \) at \( C \).

In particular, when \( K_1 \) and \( K_2 \) are both circles and their rotations are of constant velocities, we have by (4) and (5) above obtained, and (2) in the report (II)

\[
(6)_1 \quad w_{t1} = -\text{sgn}(\theta) \left( \frac{1}{a_1} - \frac{1}{a_2} \right) \left( \frac{ds}{dt} \right)^2 \cos \theta,
\]

\[
(6)_1 \quad w_{n1} = \left( \frac{1}{a_1} - \frac{1}{a_2} \right)^2 \left( \frac{ds}{dt} \right)^2 \frac{r^2}{m_1},
\]

and

\[
(6)_2 \quad w_{t2} = -\text{sgn}(\theta) \left( \frac{1}{a_2} - \frac{1}{a_1} \right) \left( \frac{ds}{dt} \right)^2 \cos \theta,
\]

\[
(6)_2 \quad w_{n2} = \left( \frac{1}{a_2} - \frac{1}{a_1} \right)^2 \left( \frac{ds}{dt} \right)^2 \frac{r^2}{m_2},
\]

and further from (6) and (6) obtain

\[
(7) \quad \frac{w_{n1}}{w_{n2}} = \frac{m_1}{m_2}.
\]

Therefore, from (4), (6) and (7) is derived the following:
Theorem 1. Given a pair of pitch circles which make rolling contact motion with constant velocity of rotation and a pair of profile curves invariably connected with those pitch circles. The velocities of sliding at any point of contact of the profile curves are proportional to the distance from the point to the pitch point corresponding to it, and the tangential components of the accelerations are proportional to the cosine of the angle between the straight line connecting the point of contact with the pitch point corresponding to it and the common tangent to the pitch curves at the pitch point. Furthermore the ratio of the normal components of the accelerations is equal to the ratio of the radii of curvature of the profile curves.

§ 2. The types of profile curves.

When we particularly adopt the rolling curve $K_r$ as one of the pitch curves $K_1$ and $K_2$, we have from (4)

$$ \frac{dp_1}{dt} = \left( \frac{1}{a_r} - \frac{1}{a_1} \right) r \frac{ds}{dt} \quad \text{and} \quad \frac{dp_2}{dt} = \left( \frac{1}{a_r} - \frac{1}{a_2} \right) r \frac{ds}{dt}.$$

Consequently we have the following relations (9) between the arc length $ds$ of the pitch curve $K$ and the arc length $dp$ of the profile curve $F$ corresponding to it:

$$ dp_1 = \left( \frac{1}{a_r} - \frac{1}{a_1} \right) r ds, \quad dp_2 = \left( \frac{1}{a_r} - \frac{1}{a_2} \right) r ds.$$

In accordance with (9) we can derive the following theorem concerning the types of roulettes, namely, of profile curves.

Theorem 2. Let a curve $K_r$ with the natural equation $a_r=a_r(s)$ roll without sliding along a curve $K$ with the natural equation $a=a(s)$. In the range of $s$, where the curvature $\frac{1}{a_r(s)}$ of $K_r$ is larger than $K$’s; $\frac{1}{a_r(s)} > \frac{1}{a(s)}$, the roulette $F$ drawn by a point $C$ fixed at $K_r$ is of positive type as far as the point $C$ exists on the left side of the common tangent of $K_r$ and $K$ at the common pitch point, and negative type as far as $C$ exists on the right side. In the range, where $\frac{1}{a_r(s)} < \frac{1}{a(s)}$, the converse holds.

Moreover, we have the following theorem concerning the assortment of the types of a pair of profile curves.

Theorem 3. Let the natural equations of a pair of profile curves $K_1$ and $K_2$ and rolling curve $K_r$ be $a_1=a_1(s)$, $a_2=a_2(s)$ and $a_r=a_r(s)$ respectively. In the range of $s$, where the curvature $\frac{1}{a_1(s)}$ of $K_r$ is larger or smaller than both of the radii of curvature $\frac{1}{a_1(s)}$ and $\frac{1}{a_2(s)}$ of $K_1$ and $K_2$, in other
words, both $K_1$ and $K_2$ exist on one side of $K_r$, the same type parts of $F_1$ and $F_2$ are in mesh, and in the range, where $\frac{1}{a_1(s)}$ exists between $\frac{1}{a_1(s)}$ and $\frac{1}{a_2(s)}$, that is, $K_r$ exists between $K_1$ and $K_2$, the different type parts of $F_1$ and $F_2$ are in mesh.

By Theorem 5 in the report (II), $a_r$, the radius of curvature $K_r$, is equal to the length of the segment cutten off by the normal to the path of contact $P$ on the perpendicular $P_0N_0$ to the initial line $P_0T_0$ at the pole $P_0$. Consequently, when both the pitch curves $K_1$ and $K_2$ are circles, we can state Theorem 3 in the following manner.

**Theorem 4.** Given a pair of pitch circles $O_1$ and $O_2$ touching at a point $P_0$ and a path of contact $P$ settled at their common tangent $P_0T_0$. Let $M$ be the point at which a normal to $P$ intersects the straight line $O_1O_2$ connecting the centers of circles $O_1$ and $O_2$. As far as $M$ exists on the one of the two parts of the center line $O_1O_2$ divided by the two points $O_1$ and $O_2$, on which part the point $P_0$ is contained, are in mesh the same type parts of the profile curves corresponding to $P$. When $M$ exists on the part not containing $P_0$, are in mesh the different type parts of the profile curves are in mesh.

§ 3. Specific slidings of profile curves.

Now we proceed to define

\[
\sigma_1 = \frac{dP_2 - dP_1}{dP_1}, \quad \sigma_2 = \frac{dP_1 - dP_2}{dP_2}
\]

and name them respectively the specific slidings of the profile curves $F_1$ and $F_2$ (at the point $C$ on $F_2$ and $F_1$). When the same type parts of $F_1$ and $F_2$ are in mesh, then $\sigma_1$ and $\sigma_2$ have different signs. When the different type parts are in mesh, $\sigma_1$ and $\sigma_2$ have the same sign and are both negative. In addition, from (10) evidently follows the relation:

\[
\frac{1}{\sigma_1} + \frac{1}{\sigma_2} = -1.
\]

From (1) and (10) we have

\[
\sigma_1 = v_1 \frac{dP_1}{dl}, \quad \sigma_2 = v_2 \frac{dP_2}{dl},
\]

Substituting (4) and (8) into (12), we have

\[
\sigma_1 = \sigma_1(s) = \frac{1}{a_1(s)} - \frac{1}{a_2(s)} - \frac{1}{a_2(s)} - \frac{1}{a_1(s)}, \quad \sigma_2 = \sigma_2(s) = \frac{1}{a_2(s)} - \frac{1}{a_1(s)} - \frac{1}{a_1(s)} - \frac{1}{a_2(s)}
\]
From this follows immediately the fact:

For any profile curves with the same pitch curves and rolling curve the specific slidings at the points of contact corresponding to the same pitch point are all equal, wherever a drawing point is set at the rolling curve.

When the equation \( r = f(s) \) of the profile curve \( F \) is given, we have, substituting Equation (8) in the report (II) into the above equation (13),

\[
\sigma_1 = \sigma_1(s) = \left( \frac{1}{a_1(s)} - \frac{1}{a_2(s)} \right) \left( \frac{1 - \left[ f'(s) \right]^2}{f(s) \sqrt{1 - \left[ f'(s) \right]^2}} - \frac{1}{a_1(s)} \right),
\]

(14)

\[
\sigma_2 = \sigma_2(s) = \left( \frac{1}{a_2(s)} - \frac{1}{a_1(s)} \right) \left( \frac{1 - \left[ f'(s) \right]^2}{f(s) \sqrt{1 - \left[ f'(s) \right]^2}} - \frac{1}{a_2(s)} \right).
\]

When the equation \( r = g(\theta) \) of the path of contact is given, we have, substituting Equation (14) in the report (II) into the above equation (13)

\[
\sigma_1 = \sigma_1(\theta) = \frac{1}{\sin \theta + \frac{\cos \theta}{g(\theta)'} \frac{1}{a_1(s(\theta))}},
\]

(15)

\[
\sigma_2 = \sigma_2(\theta) = \frac{1}{\sin \theta + \frac{\cos \theta}{g(\theta)'} \frac{1}{a_2(s(\theta))}},
\]

where \( s(\theta) = \int \frac{g(\theta)'}{\cos \theta} d\theta \).

§ 4. The radii of curvature of profile curves.

We denote by \( m \) the radius of curvature of the profile curve \( F \) at the point \( C \) on \( F \). As we have defined, the infinitesimal arc \( dp \) of \( F \) is oriented, according to this orientation we give \( m \) a positive or negative sign by the method we have already explained at the beginning of the report (II). Then we have

\[
\frac{\pm m}{r \pm m} = \frac{dp}{ds |\sin \theta|}, \text{ namely, } \frac{dp}{ds} = \frac{\pm m}{r \pm m} |\sin \theta|,
\]

where we take, from double signs \( \pm \) before \( m \), + if \( F \) is of positive type and - if \( F \) of negative type. It follows from (8) and (16)

\[
\frac{1}{a_r} - \frac{1}{a} = \left( \frac{1}{r} - \frac{1}{r \pm m} \right) \sin \theta.
\]

This is the formula of Savary concerning the radius curvature of a roulette drawn by a point fixed at a curve \( K_r \) when \( K_r \) rolls without sliding along a curve \( K \).
From (17) we can derive the relation between the radii of curvature $m_1$ and $m_2$ of a pair of profile curves $F_1$ and $F_2$ at a point of contact:

\[(18) \quad \frac{1}{a_1} - \frac{1}{a_2} = \left( \frac{1}{r + m_1} - \frac{1}{r + m_2} \right) \sin \theta,\]

where out of the double signs before $m_1$ and $m_2$ — in total four signs —, we assort the same two if $F_1$ and $F_2$ are of the same type, and the different two if $F_1$ and $F_2$ are of the different types.

Substituting (17) and (18) into (13) we have

\[(19) \quad \frac{1}{r + m_1} - \frac{1}{r + m_2} = \frac{1}{1} - \frac{1}{1} - \frac{1}{1} - \frac{1}{1} .\]

Now we shall consider a profile curve $F$ and a parallel profile curve $F^*$ with the distance $a$ from $F$. Denote by $K_r$ and $K_r^*$ respectively the rolling curves for $F$ and $F^*$, and let $a_r = a_r(s)$ and $a_r^* = a_r^*(s)$ be the natural equations of $K_r$ and $K_r^*$ respectively. By Equation (5) in the report (II) we can derive

\[(20) \quad \frac{1}{a_r} - \frac{1}{a_r^*} = \left( \frac{1}{r + a} - \frac{1}{r} \right) \sin \theta.\]

Comparing (20) with (17), we obtain the following

**Theorem 5.** Let $F$ and $F^*$ be two parallel profile curves invariably connected with a pitch curve $K$, and let $K_r$, $K_r^*$ and $C$, $C^*$ be the rolling curves and drawing points for $F$ and $F^*$ respectively. The roulette drawn by $C$, when $K_r$ rolls without sliding along $K_r^*$, is a circular arc with $C^*$ as its center and the distance of $F$ and $F^*$ as its radius.

If we denote this circular arc by $F_r^*$, then by the Camus' theorem in the report (I) the curve $F_r^*$ and $F$ are a pair of profile curves having the curves $K_r^*$ and $K$ as a pair of pitch curves.

In conclusion I express my hearty thanks to Prof. T. Kubota, M.J.A. who has given me kind guidance for the researches, and in addition I am obliged to him for communicating this paper.