113. *Permanent Strain in Tube-Wall Yielding under Internal Pressure.*

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1. Fundamentals.

In order to give simple mathematical expressions to the permanent strain caused in tube-wall of circular form yielding under internal pressure, we put down the basic points of the calculation in the following:

- (I) the permanent strain is distributed evenly about the axis of tube, and it is expressible by the radial and tangential components, which are linear functions of the displacement-components;
- (II) the total strain (permanent strain plus elastic strain) can be expressed by the radial displacement only, though the individual strains may contain terms of the tangential displacement;
- (III) the permanent strain being caused by slipping along planes bisecting the angles between the radial and tangential directions (cp. (VI)), it is in a plane-strain-state;
- (IV) no volume change occurs by the permanent strain;
- (V) the stress depends solely on the elastic strain, being quite independent of the permanent strain;
- (VI) the stress-distribution is symmetric about the axis of tube, and the yielding takes place simply by the greatest shear stress, that is, when the difference between the tangential and radial stresses equals the yield stress.

The items (I)–(IV) characterize the strains as the object of the analysis, and (V) depicts the relation as shown by a descending branch of the stress-strain curve drawn in the plastic state of metals. The last item (VI), of which the first part follows naturally from a similar thing assumed for the strains, refers mainly to the condition of yielding expressed in general term, a more precise formulation being made subsequently.

In tensile tests of mild steel specimens, for example, the yielding starts usually at the upper yield-stress, and the stress drops to the lower yield-stress at which the permanent strain grows up to a certain limit. In mild steel tubes the amount of the stress-difference (tangential stress minus radial stress) at the moment of yielding, which begins from the inner surface, is generally greater than the

upper yield-stress found in tensile tests. This discrepancy comes, it is thought, from the stress being not uniform, since yielding is caused probably not by stress at a point but by stress-force in a certain area. Then, yielding must be discontinuous taking place layer by layer; it is taken, however, conventionally as continuous going with a peak-stress-difference equal to k say. Moreover, k is treated as constant during the transition of the plastic-elastic boundary. The stress-difference in the plastic region within the boundary is uniform; so it can be put equal to the very yield-stress and to the lower one indeed, which is written s . Let the outer and inner radii of tube be denoted by R and a , respectively. Then, the yielding starts at the internal pressure p_0 , when

$$2p_0M = k > s_0 > s, \quad M = \frac{N^2}{N^2 - 1}, \quad N = \frac{R}{a}, \quad (1)$$

where s_0 is the upper yield-stress.

2. Permanent Strain.

First, we assume the permanent strain, write the stress-components, and then try to satisfy the condition of yielding. Denoting the radial and tangential components of the permanent strain by e_r and e_t ,

$$e_r + e_t = 0 \quad (2)$$

by (III) and (IV). Taking two sets of e_r and e_t , we put

$$e_r = -\frac{c}{r^2}, \quad e_t = \frac{c}{r^2}, \quad (3)$$

and

$$e'_r = c', \quad e'_t = -c'. \quad (4)$$

The last component implies a term due to the displacement tangential to the arc $r d\varphi$, as (4) comes out by putting $u = c'r$ and $v = -2c'r\varphi$ in

$$e'_r = \frac{\partial u}{\partial r}, \quad e'_t = \frac{u}{r} + \frac{\partial v}{r \partial \varphi},$$

u and v being the displacements in the radial and tangential directions, respectively.

Writing $\varepsilon_r, \varepsilon_t$ for the components of the elastic strain, and U for the total displacement, we have

$$\frac{dU}{dr} = \varepsilon_r + e_r + e'_r, \quad \frac{U}{r} = \varepsilon_t + e_t + e'_t, \quad (5)$$

since the total displacement is purely radial by (II). According to (V), the stress-components (in plane-strain-state) are

$$\sigma_r = \frac{2G}{1-2\nu} [(1-\nu)\varepsilon_r + \nu\varepsilon_t], \quad \sigma_t = \frac{2G}{1-2\nu} [\nu\varepsilon_r + (1-\nu)\varepsilon_t],$$

with G =modulus of rigidity and ν =Poisson's ratio, that is,

$$\left. \begin{aligned} \sigma_r &= \frac{2G}{1-2\nu} \left[(1-\nu) \frac{dU}{dr} + \nu \frac{U}{r} \right] - 2G(e_r + e'_r), \\ \sigma_t &= \frac{2G}{1-2\nu} \left[\nu \frac{dU}{dr} + (1-\nu) \frac{U}{r} \right] - 2G(e_t + e'_t). \end{aligned} \right\} \quad (6)$$

Substituting (6) in the equation of equilibrium $\frac{d}{dr}(r\sigma_r) = \sigma_t$, we obtain

$$r \frac{d^2U}{dr^2} + \frac{dU}{dr} - \frac{U}{r} = \frac{1-2\nu}{1-\nu} 2c'. \quad (7)$$

The general solution of (7) is

$$U = \frac{1-2\nu}{1-\nu} c' r \ln r + \frac{k_1}{2} r + \frac{k_2}{r}, \quad (8)$$

k_1 and k_2 being constants of integration; so (6) becomes

$$\sigma_r = A - \frac{B}{r^2} + C \ln r, \quad \sigma_t = A + C + \frac{B}{r^2} + C \ln r, \quad (9)$$

where

$$A = \frac{G}{1-2\nu} k_1, \quad B = 2G(k_2 - c),$$

$$\text{and} \quad C = \frac{2G}{1-\nu} c' = E' c' \quad E' = \frac{E}{1-\nu^2}.$$

The condition of yielding in the plastic region, that is, $\sigma_t - \sigma_r = s$, is satisfied by putting $B=0$ and $C=s$ in (9), from which the elastic-strain-components are found

$$\left. \begin{aligned} E' \epsilon_r &= \frac{1-2\nu}{1-\nu} (A + C \ln r) - \frac{\nu}{1-\nu} C, \\ E' \epsilon_t &= \frac{1-2\nu}{1-\nu} (A + C \ln r) + C, \quad C=s. \end{aligned} \right\} \quad (10)$$

As we may write $\epsilon_r = \frac{\partial u_r}{\partial r}$ with u_r representing the elastic displacement in the radial direction, we have by integrating ϵ_r , and dividing by r ,

$$E' \frac{u_r}{r} = \frac{1-2\nu}{1-\nu} (A + C \ln r) - C,$$

so that

$$\epsilon_t = \frac{u_r}{r} + \frac{2C}{E'} = \frac{u_r}{r} + 2c',$$

showing that ϵ_t has the component $2c'$ due to the tangential displacement; this term neutralizes (and is neutralized by) the corresponding term $-2c'$ in e'_t ; otherwise each of these terms causes the dislocation of rotational type.

3. Stress and Strain Existing under Pressure.

Let r_i be the radius of the internal boundary in being at the internal pressure p . The outer region beyond this boundary being elastic, the stress-components are

$$\sigma'_r = A' - \frac{B'}{r^2}, \quad \sigma'_t = A' + \frac{B'}{r^2}. \quad (11)$$

For determining the unknowns A , A' , B' , p , and c , we have so many boundary conditions, namely,

$$\text{at inner boundary } r=a, \quad \sigma_r = -p, \text{ i.e., } A + C \ln a = -p; \quad (12)$$

$$\text{at int. boundary } r=r_i, \quad \sigma_r = \sigma'_r, \quad \epsilon_t + e_t + e'_t = \epsilon'_t, \quad (13), (14)$$

$$\sigma'_t - \sigma'_r = k, \text{ i.e., } \frac{2B'}{r_i^2} = k; \quad (15)$$

$$\text{at outer boundary } r=R, \quad \sigma'_r = 0, \text{ i.e., } A' = \frac{B'}{R^2} = \frac{k}{2} \frac{x^2}{N^2} \text{ with } x = \frac{r_i}{a}. \quad (16)$$

When x is given, σ'_r is found by using (15) and (16), whereas σ_r is made known by (12) except for p ; so we find by (13) that

$$p = \frac{k}{2} \left(1 - \frac{x^2}{N^2} \right) + s \ln x. \quad (17)$$

As to (14), it reminds us that the plane-strain owes to the axial stresses acting in the respective regions:

$$\sigma_{z0} = \nu(2A + s + s \ln r^2), \quad \sigma'_{z0} = \nu 2A'. \quad (18)$$

When the axial pull P acts on the tube, the axial stresses are

$$\sigma_z = \sigma_{z0} + \bar{\sigma}_z \quad \text{and} \quad \sigma'_z = \sigma'_{z0} + \bar{\sigma}_z, \quad (19)$$

respectively, with the condition that

$$\int_a^{r_i} \sigma_z r dr + \int_{r_i}^R \sigma'_z r dr = \frac{P}{2\pi}. \quad (20)$$

From (19) and (20) $\bar{\sigma}_z$ is found; the result of the calculation is

$$(N^2 - 1)\bar{\sigma}_z = q - \nu 2p, \quad q = \frac{P}{\pi a^2}. \quad (21)$$

In case of no other load than the internal pressure being applied, $q = p$ or 0 according as the ends are covered or open. As $\bar{\sigma}_z$ is common to the two regions, it has no effect on (14), in which e_t , e'_t , and ϵ_t are given by (3), (4), and (10), while ϵ'_t is obtained by means of (15) and (16). After intermediate calculation we find that

$$c = \frac{k}{E'} r_i^2. \quad (22)$$

In virtue of (22), the permanent strain (3) at any point, $a \leq r \leq r_i$, becomes determinate.

The stress and strain (tangential) are now as follows:

$$\left. \begin{aligned} r < r_i, \quad \sigma_r = s \ln \frac{r}{a} - p, \quad \sigma_t = s \ln \frac{r}{a} + s - p, \quad \sigma_t - \sigma_r = s; \\ r > r_i, \quad \sigma'_r = -\frac{k}{2} \frac{x^2}{N^2} \left(\frac{R^2}{r^2} - 1 \right), \quad \sigma'_t = \frac{k}{2} \frac{x^2}{N^2} \left(\frac{R^2}{r^2} + 1 \right), \quad \sigma'_t - \sigma'_r = k \frac{r_i^2}{r^2}. \end{aligned} \right\} \quad (23)$$

$$\left. \begin{aligned} r < r_i, \quad E' \left[\epsilon_t + e_t + e'_t - \frac{\nu \bar{\sigma}_z}{E} \right] = k \frac{r_i^2}{r^2} + \frac{1-2\nu}{1-\nu} \left(s \ln \frac{r}{a} - p \right) - \frac{\nu}{1-\nu^2} \bar{\sigma}_z; \\ r > r_i, \quad E' \left[\epsilon'_t - \frac{\nu \bar{\sigma}_z}{E} \right] = \frac{1-2\nu}{1-\nu} \frac{k}{2} \frac{x^2}{N^2} + \frac{1}{1-\nu} \frac{k}{2} \frac{r_i^2}{r^2} - \frac{\nu}{1-\nu^2} \bar{\sigma}_z. \end{aligned} \right\} \quad (24)$$

Particularly, at the inner and outer surfaces, the right sides of (24) become

$$r=a, \quad kx^2 - p \frac{1-2\nu}{1-\nu} - \frac{\nu}{1-\nu^2} \bar{\sigma}_z; \quad r=R, \quad k \frac{x^2}{N^2} - \frac{\nu}{1-\nu^2} \bar{\sigma}_z. \quad (25)$$

4. Residual Stress and Strain.

On unloading, the stresses (23) cannot vanish completely. What vanishes is the stress existing irrespectively of the permanent strain, that is,

$$\sigma_{r0} = \frac{p}{N^2 - 1} \left(1 - \frac{R^2}{r^2} \right), \quad \sigma_{t0} = \frac{p}{N^2 - 1} \left(1 + \frac{R^2}{r^2} \right). \quad (26)$$

The corresponding strain (tangential) is

$$E' \epsilon_{t0} = \frac{1}{1-\nu^2} \frac{p}{N^2 - 1} \left[(1+\nu) \frac{R^2}{r^2} + 1 - \nu \right] - \frac{\nu}{1-\nu^2} \frac{q}{N^2 - 1}. \quad (27)$$

The residual stress may be found from (23) and (26). Putting $2\tau^* = \sigma_t - \sigma_r - (\sigma_{t0} - \sigma_{r0})$, it is seen, as $2p_0M = k$ by (1), that

$$\begin{aligned} r=a \quad 2\tau^* &= s - 2pM = s - k \frac{p}{p_0}; \\ r=r_i-0, \quad 2\tau^* &= s - \frac{k}{x^2} \frac{p}{p_0}; \quad r=r_i+0, \quad 2\tau^* = k - \frac{k}{x^2} \frac{p}{p_0}. \end{aligned} \quad (23a, b, c)$$

Of these expressions, (23a) is negative (under special conditions $-2\tau^*$ may become so large as to cause yielding in compression), and ($\sim c$) is generally positive, since it can be shown that $x^2 \geq \frac{p}{p_0}$. As to ($\sim b$), it is negative for $x=1$, and it increases with x , being certainly positive for $x=N$. Besides, the discontinuity of σ_t and σ'_t at $r=r_i$, namely, by the amount $(k-s)$ is seen in the difference between ($\sim b$) and ($\sim c$).

Similarly, the residual strain ϵ_t^* is found from (25) and (27). The calculation shows that

$$\left. \begin{aligned} r=a, \quad E' \epsilon_t^* &= kx^2 - 2pM = k \left(x^2 - \frac{p}{p_0} \right), \\ r=R, \quad E' \epsilon_t^* &= k \frac{x^2}{N^2} - \frac{2pM}{N^2} = \frac{k}{N^2} \left(x^2 - \frac{p}{p_0} \right), \end{aligned} \right\} \quad (29)$$

both being positive.

When the wall yields completely, $x=N$ and $c=k\frac{R^2}{E'}$. With further application of the pressure, $p=s\ln N$, then acting, c may increase beyond the above value, as the elastic constraint is off, so far as the amount is still moderate to be compatible with the premise implied by e_r and e_t in (3), which are linear functions of the radial displacement $\frac{c}{r}$. In contrast with this, c' remains constant so long as s is invariable, and the residual stress too, being given by the following expressions — the same expressions as for a ring-sector bent by couples:

$$\left. \begin{aligned} \sigma_r^* &= \sigma_r - \sigma_{r0} = s \left[\left(\frac{R^2}{r^2} - 1 \right) \frac{\ln N}{N^2 - 1} - \ln \frac{R}{r} \right], \\ \sigma_t^* &= \sigma_t - \sigma_{t0} = s \left[1 - \ln \frac{R}{r} - \left(\frac{R^2}{r^2} + 1 \right) \frac{\ln N}{N^2 - 1} \right]. \end{aligned} \right\} \quad (30)$$

Concluding Remarks.— Looking back upon the two sets of the permanent strains stated above, we see that one grows freely giving rise to no residual stress, and the other is invariable, being combined with the elastic strain and stress. In view of the stress-strain relation being independent of the permanent strain, it is natural that the magnitude of the permanent strain is indefinite; it becomes definite only when subjected to a fixed relation with the elastic strain. This being so, if a tube stored with the residual stress (30) be cut radially across the wall in one place so as to be made free from the stress, the cut surfaces will open as wide as was strained elastically, that is, by an angle $2c' \cdot 2\pi$ with the necessary increase of the radius. This inference may be submitted to the decision by experience.

Further, be it noted that the possibility of the permanent strain arising in the axial direction was set aside in the foregoing calculation, the axial stress being assumed to be intermediate between the radial and tangential stresses. This is the case for the amount of q kept below a certain limit. If q is positive, a lower bound of this limit can be examined by taking the critical value of the difference between the axial and radial stresses equal to s ; in doing this, use may be made of (19) for finding the axial stress.

Lastly, I wish to record with gratitude that Professor J. Takenaka put unpublished results of his experiment on tubes at my disposal, so that I could refer to them freely for confirming some points of the fundamentals. Thanks are due also to Professors T. Isibasi and K. Washizu for the help in looking over some works of the recent publication, which task has induced me to think of this note being probably not superfluous to be put into print.