112. On Completeness of Uniform Spaces

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Let $R$ be an abstract space. For a system of mappings $a_{\lambda}$ of $R$ into uniform spaces $S_{\lambda}(\lambda \in \Lambda)$, the weakest uniformity on $R$ for which all $a_{\lambda}(\lambda \in \Lambda)$ are uniformly continuous, is called the weak uniformity of $R$ by $a_{\lambda}(\lambda \in \Lambda)$. Concerning the completeness of the weak uniformity we have

Theorem I. Let the uniformities $U_{\lambda}$ of $S_{\lambda}(\lambda \in \Lambda)$ be separative and complete. In order that the weak uniformity of $R$ by a system of mappings $a_{\lambda}$ of $R$ into $S_{\lambda}(\lambda \in \Lambda)$ be complete, it is necessary and sufficient that for a system of points $x_{\lambda} \in S_{\lambda}(\lambda \in \Lambda)$ if

$$\prod_{\lambda=1}^{n} a_{\lambda}^{-1}(U_{\lambda}(x_{\lambda})) = 0$$

for every finite number of elements $\lambda, \in \Lambda$ and $U_{\lambda}, U_{\lambda'} \in U_{\lambda}(\nu=1,2,\ldots,n)$, then we can find a point $x \in R$ for which $a_{\lambda}(x) = x_{\lambda}$ for every $\lambda \in \Lambda$.

The purpose of this paper is to give some generalization of this Theorem I and its applications.

I

For a uniform space $R$ with uniformity $\mathfrak{U}$, a system of mappings $a_{\gamma}(\gamma \in \Gamma)$ of $R$ into a uniform space $S$ with uniformity $\mathfrak{U}$ is said to be equi-continuous, if for any $U \in \mathfrak{U}$ we can find $V \in \mathfrak{U}$ such that

$$a_{\gamma}(V(x)) \subset U(a_{\gamma}(x))$$

for every $x \in R$ and $\gamma \in \Gamma$.

With this definition we have

Theorem II. Let the uniformity $U_{\lambda}$ of $S_{\lambda}(\lambda \in \Lambda)$ be separative and complete. For a double system of mappings $a_{\gamma,\lambda}$ of an abstract space $R$ into $S_{\lambda}(\gamma \in \Gamma, \lambda \in \Lambda)$, there exists the weakest uniformity on $R$ for which $a_{\gamma,\lambda}(\gamma \in \Gamma, \lambda \in \Lambda)$ is equi-continuous for every $\lambda \in \Lambda$, and in order that this uniformity on $R$ be complete, it is necessary and sufficient that for a system of points $x_{\gamma,\lambda} \in S_{\lambda}(\gamma \in \Gamma, \lambda \in \Lambda)$ if

$$\prod_{\gamma=1}^{n} \prod_{\lambda=1}^{m} a_{\gamma,\lambda}^{-1}(U_{\lambda}(x_{\gamma,\lambda})) = 0$$

for every finite number of elements $\lambda, \in \Lambda$ and $U_{\lambda}, U_{\lambda'} \in U_{\lambda}(\nu=1,2,\ldots,n)$, then we can find a point $x \in R$ such that

$x_{\gamma,\lambda} = a_{\gamma,\lambda}(x)$

for all $\gamma \in \Gamma, \lambda \in \Lambda$.

In order to prove this Theorem II, we shall define power of a uniformity. Let $S$ be a uniform space with uniformity $\mathfrak{U}$. For another abstract space $A$, considering every system $x_\lambda \in S (\lambda \in A)$ a point $(x_\lambda)_{\lambda \in A}$, we obtain a space, which is called the power of $S$ by $A$ and denoted by $S^A$. For each $U \in \mathfrak{U}$, putting 
\[ U^A(x_\lambda)_{\lambda \in A} = \{(y_\lambda)_{\lambda \in A} : y_\lambda \in U(x_\lambda) \text{ for every } \lambda \in A\}, \]
we obtain a connector $U^A$ in $S^A$. Furthermore we see easily that there exists uniquely a uniformity on $S^A$ of which $U^A(U \in \mathfrak{U})$ is a basis. This uniformity on $S^A$ is called the power of $\mathfrak{U}$ by $A$ and denoted by $\mathfrak{U}^A$. With this definition we can prove easily that if $\mathfrak{U}$ is separative, then $\mathfrak{U}^A$ also is separative; and if $\mathfrak{U}$ is complete, then $\mathfrak{U}^A$ also is complete.

For a system of mappings $a_\lambda (\lambda \in A)$ of a uniform space $R$ into a uniform space $S$ with uniformity $\mathfrak{U}$, it is evident by definition that $a_\lambda (\lambda \in A)$ is equi-continuous if and only if the mapping $a$ of $R$ into the power $S^A$ with uniformity $\mathfrak{U}^A$:
\[ a(x) = (a_\lambda (x))_{\lambda \in A} \in S^A \quad (x \in R) \]
is uniformly continuous. Therefore for a system of mappings $a_{\tau, \lambda}$ of an abstract space $R$ into uniform spaces $S_\tau (\gamma \in \tau', \lambda \in A)$, the weak uniformity of $R$ by the system of mappings $a_\lambda$ of $R$ into the uniform spaces $S_\tau^\lambda (\lambda \in A)$:
\[ a_\lambda (x) = (a_{\tau, \lambda}(x))_{\tau \in \tau'} \in S_\tau^\lambda \quad (x \in R) \]
is the weakest uniformity on $R$ for which $a_{\tau, \lambda} (\gamma \in \tau', \lambda \in A)$ is equi-continuous for every $\lambda \in A$. Therefore we conclude Theorem II immediately from Theorem I.

In Theorem II, if all uniform spaces $S_\lambda (\lambda \in A)$ coincide with a complete separative uniform space $S$ with uniformity $\mathfrak{U}$, and for a system of points $x_{\gamma, \lambda} \in S (\gamma \in \tau', \lambda \in A)$ if
\[ \prod_{\nu=1}^n \prod_{\tau \in \tau_\nu} a_{\tau, \lambda}^{-1} (U_\nu (x_{\tau, \lambda})) = 0 \]
for every finite number of elements $\lambda_0 \in A$ and $U_\nu \in \mathfrak{U}(\nu=1,2,\ldots,n)$, then $a_{\tau, \lambda} = a_{\gamma, \lambda}$ implies $x_{\gamma, \lambda} = x_{\tau, \lambda}$, because $\mathfrak{U}$ is separative by assumption. Therefore we conclude from Theorem II

**Theorem III.** Let $a_\lambda (\lambda \in A)$ be a system of mappings of an abstract space $R$ into a uniform space $S$ with a complete separative uniformity $\mathfrak{U}$. For a system of subsets $\Lambda_\gamma \subset A (\gamma \in \Gamma)$ there exists the weakest uniformity on $R$ for which $a_\lambda (\lambda \in \Lambda_\gamma)$ is equi-continuous for every $\gamma \in \Gamma$, and if $\Lambda = \sum_{\gamma \in \Gamma} \Lambda_\gamma$ and for any $\gamma_1, \gamma_2 \in \Gamma$ we can find an element $\gamma \in \Gamma$ such that $\Lambda_{\gamma_1} \cap \Lambda_{\gamma_2} \subset \Lambda_\gamma$, then in order that this weakest uniformity on $R$ be complete, it is necessary and sufficient that for a system of points $x_\lambda \in S (\lambda \in A)$ if
\[ \prod_{\gamma \in \Gamma} a_\lambda^{-1} (U (x_\lambda)) = 0 \]
for every \( \gamma \in \Gamma \) and \( U \in \Omega \), then we can find a point \( x \in R \) such that 
\[ \alpha_{\lambda}(x) = x_\lambda \] 
for every \( \lambda \in \Lambda \).

Let \( \mathfrak{A} \) be the totality of mappings of an abstract space \( R \) into a uniform space \( S \) which a complete separative uniformity \( \Omega \). Every point \( x \in R \) may be considered a mapping of \( \mathfrak{A} \) into \( S \) as \( a(x) \in S(a \in \mathfrak{A}) \). For a system of subsets \( R_\lambda \subseteq R(\lambda \in \Lambda) \) there exists by Theorem III the weakest uniformity on \( \mathfrak{A} \) for which \( R_\lambda \) is equi-continuous as a system of mappings for every \( \lambda \in \Lambda \). This weakest uniformity on \( \mathfrak{A} \) is complete \(^3\), because for any system of points \( y_\lambda \in S(x \in R) \) there exists obviously \( a \in \mathfrak{A} \) for which \( a(x) = y_\lambda \) for every \( x \in R \).

A mapping \( a \) of \( R \) into \( S \) is said to be bounded in a subset \( R_0 \subseteq R \), if the image \( a(R_0) \) is a bounded set \(^3\) of \( S \). For a uniformity on \( \mathfrak{A} \) if \( R_\lambda \) is equi-continuous as a system of mappings of \( \mathfrak{A} \) into \( S \), then we see easily by definition that every convergence by a Cauchy system in \( \mathfrak{A} \) is a uniform convergence as mappings of \( R_\lambda \) into \( S \). Therefore on the totality of those mappings of \( R \) into \( S \) which are bounded in \( R_\lambda \) for every \( \lambda \in \Lambda \), the weakest uniformity for which \( R_\lambda \) is equi-continuous for every \( \lambda \in \Lambda \), is complete. We conclude further that if \( R \) is a topological space, then on the totality of those mappings of \( R \) into \( S \) which are continuous in \( R_\lambda \) by the relative topology for every \( \lambda \in \Lambda \), the weakest uniformity for which \( R_\lambda \) is equi-continuous for every \( \lambda \in \Lambda \), is complete. Furthermore we obtain likewise that if \( R \) is a uniform space, then on the totality of those mappings of \( R \) into \( S \) which are uniformly continuous in \( R_\lambda \) by the relative uniformity for every \( \lambda \in \Lambda \), the weakest uniformity for which \( R_\lambda \) is equi-continuous for every \( \lambda \in \Lambda \), is complete.

II

Let \( R \) be a linear space and \( S \) a linear topological space with linear topology \( \mathfrak{B} \). A system of linear operators \( T_\lambda(\lambda \in \Lambda) \) on \( R \) into \( S \) is said to be bounded, if the system \( T_\lambda x(\lambda \in \Lambda) \) is a bounded set of \( S \) for every \( x \in R \). For a bounded system of linear operators \( T_\lambda(\lambda \in \Lambda) \) on \( R \) into \( S \) we see easily that there exists uniquely a linear topology on \( R \) of which 
\[ \prod_{x \in \mathfrak{B}} \{ x : T_\lambda x \in V \} \quad (V \in \mathfrak{B}) \]
is a basis. Furthermore we see easily that the induced uniformity from this linear topology on \( R \) is the weakest uniformity on \( R \) for

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2) This fact was proved by N. Bourbaki, Topologie générale, Vol. 3, Chapter 10, espaces fonctionnels, Paris (1949).
3) TLTS § 32.
which \( T_\gamma(\lambda \in \Lambda) \) is equi-continuous. This linear topology on \( R \) is obviously convex, if \( B \) is convex. Therefore, recalling Theorem 5 in TLTS § 55, we obtain by Theorem III

**Theorem IV.** Let \( T_\gamma(\lambda \in \Lambda) \) be a system of linear operators on a linear space \( R \) into a linear topological space \( S \) with a complete separative linear topology \( B \). For a system of subsets \( \Lambda \), if \( T_\gamma(\lambda \in \Lambda) \) is a bounded system for every \( \gamma \in \Gamma \), then there exists uniquely a linear topology on \( R \) whose induced uniformity on \( R \) is the weakest uniformity for which \( T_\gamma(\lambda \in \Lambda) \) is equi-continuous for every \( \gamma \in \Gamma \). Furthermore if \( \Lambda = \sum_{\tau \in \mathcal{P}} \Lambda_{\tau} \) and for any \( \gamma_1, \gamma_2 \in \Gamma \) we can find an element \( \gamma \in \Gamma \) such that \( \Lambda_{\gamma_1} \cup \Lambda_{\gamma_2} < \Lambda_\gamma \), then in order that this linear topology on \( R \) be complete, it is necessary and sufficient that for a system of elements \( x_\lambda \in S(\lambda \in \Lambda) \) if

\[
\prod_{\lambda \in \Lambda} \{ x : T_\lambda x \in V + x_\lambda \} = 0
\]

for every \( \gamma \in \Gamma \) and \( V \in B \), then we can find an element \( x \in R \) for which \( T_\lambda x = x_\lambda \) for every \( \lambda \in \Lambda \).

Let \( R \) be an abstract space and \( S \) a linear topological space with a complete separative linear topology \( B \). For a subset \( R_\lambda \subset R \) a mapping \( a \) of \( R \) into \( S \) is said to be bounded if the image \( a(R_\lambda) \) is a bounded set of \( S \). For a system of subsets \( R_\lambda \subset R(\lambda \in \Lambda) \), denoting by \( \mathcal{A} \) the totality of those mappings of \( R \) into \( S \) which are bounded in \( R_\lambda \), we obtain a linear space \( \mathcal{A} \), defining

\[(\alpha a + \beta b)(x) = \alpha a(x) + \beta b(x) \quad (x \in R)\]

for every \( a, b \in \mathcal{A} \) and real numbers \( \alpha, \beta \). Furthermore every point \( x \in R \) may be considered a linear operator on \( R \) into \( S \) as \( a(x) \in S \) \((a \in \mathcal{A})\) and \( R_\lambda \) is a bounded system of linear operators for every \( \lambda \in \Lambda \). For a system of elements \( x_\gamma \in S(\gamma \in \sum_{\lambda \in \Lambda} R_\lambda) \) if

\[
\prod_{\gamma \in \sum_{\lambda \in \Lambda} R_\lambda} \{ a : a(y) \in V + x_\gamma \} = 0
\]

for every \( \lambda \in \Lambda \) and \( V \in B \), then \( x_\gamma(y \in R_\lambda) \) is a bounded set of \( S \) for every \( \lambda \in \Lambda \), and hence putting \( a_0(y) = x_\gamma \), \( y \in \sum_{\lambda \in \Lambda} R_\lambda \) and \( a_0(y) = 0 \) for every other point \( y \), we have \( a \in \mathcal{A} \). Therefore we obtain by Theorem IV

**Theorem V.** Let \( R \) be an abstract space and \( S \) a linear topological space with a complete separative linear topology \( B \). For a system of subsets \( R_\lambda \subset R(\lambda \in \Lambda) \) such that for any \( \lambda_1, \lambda_2 \in \Lambda \) we can find an element \( \lambda \in \Lambda \) for which \( R_{\lambda_1} \cup R_{\lambda_2} < R_\lambda \), denoting by \( \mathcal{A} \) the totality of those mappings of \( R \) into \( S \) which are bounded in \( R_\lambda \), for every \( \lambda \in \Lambda \), we obtain a complete linear topological space \( \mathcal{A} \) such that

\[(\alpha a + \beta b)(x) = \alpha a(x) + \beta b(x) \quad (x \in R)\]

for every \( a, b \in \mathcal{A} \) and real numbers \( \alpha, \beta \), and

\[
\{ a : a(R_\lambda) < V \} \quad (\lambda \in \Lambda, V \in B)
\]
is a basis of \( \mathcal{U} \). Furthermore if \( \mathcal{V} \) is convex, then \( \mathcal{U} \) also is convex.

If \( R \) is a linear space and for any \( \lambda_1, \lambda_2 \in \Lambda \) and real numbers \( \alpha, \beta \) we can find an element \( \lambda \in \Lambda \) such that \( \alpha R_{\lambda_1} \times \beta R_{\lambda_2} \subseteq R_{\lambda} \), then, denoting by \( \mathcal{U} \) the totality of those linear operators on \( R \) into \( S \) which are bounded in \( R_{\lambda} \) for every \( \lambda \in \Lambda \) we see easily that
\[
\Pi \{ a : a(y) \in V + x_{\lambda} \} = 0
\]
for every \( \lambda \in \Lambda \) and \( V \in \mathcal{V} \) implies
\[
x_{\alpha y_1 + \beta y_2} = \alpha x_{y_1} + \beta x_{y_2}
\]
for every \( y_1, y_2 \in \sum \lambda R_{\lambda} \) and real numbers \( \alpha, \beta \). Therefore we obtain further

**Theorem VI.** Let \( R \) be a linear space and \( S \) a linear topological space with a complete separative linear topology \( \mathcal{V} \). For a system of subsets \( R_{\lambda} \subseteq R(\lambda \in \Lambda) \) such that for any \( \lambda_1, \lambda_2 \in \Lambda \) and real numbers \( \alpha, \beta \) we can find an element \( \lambda \in \Lambda \) such that \( \alpha R_{\lambda_1} \times \beta R_{\lambda_2} \subseteq R_{\lambda} \), denoting by \( \mathcal{X} \) the totality of those linear operators on \( R \) into \( S \) which are bounded in \( R_{\lambda} \) for every \( \lambda \in \Lambda \), we obtain a complete linear topological space \( \mathcal{X} \) such that
\[
\{ T : TR_{\lambda} \subseteq V \} \quad (\lambda \in \Lambda, V \in \mathcal{V})
\]
is a basis of \( \mathcal{X} \). Furthermore if \( \mathcal{V} \) is convex, then \( \mathcal{X} \) also is convex.

This Theorem VI is a generalization of Theorems 1 and 3 in TLTS §67.