III. Simple mother-descendants combinations

1. Mother-child-$\nu$th descendant combination

We designate, in general, by
\[
\pi_{\mu\nu}(a\beta; \xi_1, \xi_2) = A_{\mu\nu}(a\beta; \xi_1, \xi_2)
\]
the probability of a combination consisting of an individual $A_{\mu\nu}$ and its $\mu$th and $\nu$th collateral descendants $A_{\xi_1\xi_2}$ and $A_{\xi_3\xi_4}$, respectively, originated from the same spouse of $A_{\mu\nu}$.

Three systems will be distinguished according to $\mu = \nu = 1$, $\mu = 1 < \nu$ or $\mu > 1 = \nu$, and $\mu > 1 = \nu$. The lowest system has already been treated as the probability of mother-children combination
\[
\pi(a\beta; \xi_1, \xi_2) = \overline{A}_{\mu\nu}(a\beta; \xi_1, \xi_2) \quad (k = 1).
\]
In the present section we consider the second system while the last system will be postponed into the next section.

Now, based on an evident quasi-symmetry relation
\[
\pi_{\mu\nu}(a\beta; \xi_1, \xi_2) = \pi_{\nu\mu}(a\beta; \xi_1, \xi_2),
\]
it suffices to deal with the former of the second system. The reduced probability $\kappa_1$ is then defined by a recurrence equation
\[
\kappa_1(a\beta; \xi_1, \xi_2) = \sum \kappa(a\beta; \xi_1, ab)\kappa_{1-1}(ab; \xi_2).
\]
It is shown that the probability is expressed by the formula
\[
\kappa_1(a\beta; \xi_1, \xi_2) = \kappa(a\beta; \xi_1, \xi_2).\overline{A}_{\mu\nu} + 2^{-n}W(a\beta; \xi_1, \xi_2).
\]
The quantity $W(a\beta; \xi_1, \xi_2)$ in the residual term evidently vanishes out unless $A_{\xi_1\xi_2}$ possesses at least a gene in common with $A_{\mu\nu}$, and its values are given as follows; cf. also a remark stated at the end of I, §1:

\[
\begin{align*}
W(ii; ii, ii) & = 3i^2(1-i), \\
W(ii; ii, ig) & = 3ig(1-2i), \\
W(ii; ii, gf) & = -6ifg, \\
W(ii; ik, ig) & = k(i+2k-6ik), \\
W(ii; ik, kg) & = kg(1-6k), \\
W(ii; ik, gg) & = -3kg^2.
\end{align*}
\]

1) Cf. a previous paper: IV. Mother-child combinations. 27 (1951), 587-620.
The proof of the formula is performed by induction by directly verifying an identity

$$\sum W(\alpha \beta; \xi_1 \eta_1, ab)\kappa(\alpha \beta; \xi_1 \eta_1, ab)Q(ab; \xi_2 \eta_2) = \frac{1}{2} W(\alpha \beta; \xi_1 \eta_1, \xi_2 \eta_2).$$

It is noted that the quantity $W$ satisfies further identities

$$\sum W(\alpha \beta; \xi_1, ab) = 0, \quad \sum W(\alpha \beta; ab, \xi_1) = 2Q(\alpha \beta; \xi_1),$$

$$\sum A_{\alpha \beta} W(ab; \xi_1 \eta_1, \xi_2 \eta_2) = 2A_{\alpha \beta} Q(\xi_1 \eta_1; \xi_2 \eta_2).$$

2. **Mother-$\mu$th descendant-$\nu$th descendant combination**

The formula for the last generic system with $\mu, \nu > 1$ is expressed in the form

$$\kappa_\mu(\alpha \beta; \xi_1 \eta_1, \xi_2 \eta_2) = A_{\alpha \beta} A_{\xi_1 \eta_1} + 2^{-\mu+1} A_{\xi_2 \eta_2} Q(\alpha \beta; \xi_1 \eta_1)$$

$$+ 2^{-\nu+1} A_{\xi_1 \eta_1} Q(\alpha \beta; \xi_1 \eta_1) + 2^\nu T(\alpha \beta; \xi_1 \eta_1, \xi_2 \eta_2),$$

$$\lambda = \mu + \nu - 1,$$

where the values of $T$ are as follows; cf. a remark stated at the end of I, §1:

$$T(ii; ii, ii) = i^2(1-i)(2-2i), \quad T(ii; ii, ig) = ig(1-2i)(2-i),$$

$$T(ii; ii, gg) = -ig^2(2-i), \quad T(ii; ii, f g) = -2i f g(2-i),$$

$$T(ii; ik, ik) = k^2(2k + i^2 - 7ik + 4i^2k), \quad T(ii; ik, ik) = k^2(2k + 2i^2 + 7ik + 4i^2k),$$

$$T(ii; ik, ig) = kg(2-7i+4i^2), \quad T(ii; ik, gg) = -2kg^2(1-i),$$

$$T(ii; kk, kk) = k^2(1+k), \quad T(ii; kk, f g) = 2k^2 f g,$$

$$T(ii; hk, kk) = hk(k + k + 4hk), \quad T(ii; hk, f g) = hkg(1+4k),$$

$$T(ii; ik, ig) = \frac{i}{2} i^2(2i-3i),$
\[ T(\langle i; j \rangle; k, g) = 4hkfg; \]
\[ T(\langle i; j \rangle; i, i) = \frac{1}{2}i(i - 2j); \]
\[ T(\langle i; j \rangle; i, j) = \frac{1}{2}ij(1 - 2i + 2j); \]
\[ T(\langle i; j \rangle; ii, jj) = 2ij(1 - 2i + 2j); \]
\[ T(\langle i; j \rangle; ii, gg) = -i(1 - i); \]
\[ T(\langle i; j \rangle; jj, gg) = -i(1 - i); \]
\[ T(\langle i; j \rangle; ik, jk) = 2g(1 - 2i). \]

The proof of the formula can be performed by induction by means of a recurrence equation

\[ \kappa_{\mu}(\alpha \beta; \xi; \eta) = \sum \kappa_{\mu-1}(\alpha \beta; \xi; \eta), \]

together with the identities

\[ \sum W(\alpha \beta; ab, \xi; \eta) = \sum W(\alpha \beta; ab, \xi; \eta)Q(ab; \xi, \eta) \]
\[ = \frac{1}{2}T(\alpha \beta; \xi, \eta), \]
\[ \sum T(\alpha \beta; ab, \xi; \eta) = \sum T(\alpha \beta; ab, \xi; \eta)Q(ab; \xi, \eta) \]
\[ = \frac{1}{2}T(\alpha \beta; \xi, \eta). \]

It is noted that the quantity \( T \) satisfies, besides an evident symmetry relation \( T(\alpha \beta; \xi; \eta) = T(\alpha \beta; \xi; \eta) \), also an identity

\[ \sum T(\alpha \beta; ab, \xi; \eta) = 0, \quad \sum A_{ab}T(ab; \xi, \eta) = 2A_{\xi, \eta}Q(\xi, \eta). \]

An asymptotic behavior of \( \kappa_{\mu} \) as \( \nu (\text{or} \mu) \) tends to infinity can be readily deduced. In fact, we get a limit equation

\[ \lim_{\nu \to \infty} \kappa_{\nu}(\alpha \beta; \xi, \eta) = \kappa_{\nu}(\alpha \beta; \xi, \eta) \cdot A_{\xi, \eta}, \]

which remains valid for any \( \mu \) with \( \mu \geq 1 \).