48. Some Trigonometrical Series. XII

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1. A. Zygmund [1] has proved the following theorems.

**Theorem 1.** Let \( a(x) \) be a positive, decreasing and convex function in the interval \((0, \infty)\) such that

\[
(1) \quad a(x) \downarrow 0, \quad xa(x) \uparrow \quad \text{as } x \to \infty.
\]

Let \( a_n = a(n) \) and

\[
(2) \quad f(x) = \sum_{n=1}^{\infty} a_n \sin nx,
\]

then we have

\[
(3) \quad \frac{f(x)}{x} \sim x^{-1}a(x^{-1}) \quad \text{as } x \to 0.
\]

**Theorem 2.** Let \( a(x) \) be a positive, decreasing and convex function in the interval \((0, \infty)\), tending to zero as \( x \to \infty \). Let \( a_n = a(n) \) and suppose that

\[
(4) \quad na_n \downarrow, \quad \sum a_n = \infty.
\]

If we put

\[
(5) \quad f(x) = \sum_{n=1}^{\infty} a_n \cos nx,
\]

then we have

\[
(6) \quad f(x) \sim \int_{0}^{1/x} t \left| a'(t) \right| dt \quad \text{as } x \to 0.
\]

Omitting the second condition of \( a(x) \) in (1), we prove the following

**Theorem 3.** Let \( a(x) \) be a positive, decreasing and convex function in the interval \((0, \infty)\), tending to zero as \( x \to \infty \). Let \( a_n = a(n) \) and define \( \bar{f}(x) \) by (2), then

\[
(7) \quad \bar{f}(x) \sim x \int_{0}^{1/x} t a(t) dt \quad \text{as } x \to 0,
\]

when \( \bar{f}(x) \) is not bounded or the right side is ultimately positive.

If the second condition of (1) is satisfied, then we can easily see that (7) becomes (3).

In Theorem 2 we can replace the first condition of (4) by \( a_n \sum a_n \leq 0 \), that is,

**Theorem 4.** Let \( a(x) \) be a positive, decreasing and convex function in the interval \((0, \infty)\), tending to zero as \( x \to \infty \) and let \(-a'(t)\) be convex. Let \( a_n = a(n) \) and suppose that \( \sum a_n = \infty \). Then

\[
(8) \quad f(x) \sim \int_{0}^{1/x} t \left| a'(t) \right| dt \quad \text{as } x \to 0.
\]
Our proof of these theorems is very simple, except that the following lemma is used [2] (cf. [3]):

**Lemma.** If \( a_n \downarrow 0 \), then

\[
\bar{f}(x) = \int_0^\infty a(t) \sin xt \, dt + \bar{g}(x),
\]

\[
f(x) = \int_0^\infty a(t) \cos xt \, dt + g(x),
\]

where \( \bar{g}(x) \) and \( g(x) \) are bounded.

2. We shall prove Theorem 3. By Lemma, it is sufficient to prove that

\[
\bar{f}_1(x) = \int_0^\infty a(t) \sin xt \, dt
\]

satisfies the relation (8). We have

\[
\bar{f}_1(x) = \int_0^{\pi/2} b_n(t) \sin xt \, dt,
\]

where

\[
b_n(t) = a(t) + \sum_{k=1}^\infty (-1)^{k+1} \left[ a\left( \frac{k\pi}{x} - t \right) - a\left( \frac{k\pi}{x} + t \right) \right]
\]

\[
= \sum_{k=0}^\infty (-1)^k \left[ a\left( \frac{k\pi}{x} + t \right) + a\left( \frac{(k+1)\pi}{x} - t \right) \right].
\]

By the monotony of \( a(t) \), we get

\[
b_n(t) \leq a(t) + a\left( \frac{\pi}{x} - t \right)
\]

and by the convexity of \( a(t) \), we get

\[
b_n(t) \geq a(t).
\]

Hence

\[
\bar{f}_1(x) \geq \frac{2}{\pi} \int_0^{\pi/2} ta(t)dt \geq Ax \int_0^{1/2} ta(t)dt
\]

and

\[
\bar{f}_1(x) \leq \frac{2}{\pi} \int_0^{\pi/2} ta(t)dt + x \int_0^{\pi/2} \left( \frac{\pi}{x} - t \right) a(t)dt
\]

\[
\leq Ax \int_0^{1/2} ta(t)dt \leq Ax \int_0^{1/2} ta(t)dt.
\]

Thus we get (7).

We shall now prove the following

**Theorem 5.** Let \( a(x) \) be a positive decreasing sequence such that there is a positive constant \( c < 1 \) such that

\[
a(t) > ca(3t) \quad (t > 0).
\]

Let \( a_n = a(n) \) and define \( \bar{f}(x) \) by (2), then (7) holds when \( \bar{f}(x) \) is unbounded or the right of (7) is ultimately positive.

In the proof of Theorem 3, we did not use the convexity of \( (a_n) \) to prove (9). We shall prove (8) by the condition (10). By the monotony of \( a(t) \),
\[
\tilde{f}_1(x) \geq \int_0^{\pi/2x} a(t) \sin xt \, dt + \int_{\pi/2x}^{\pi} a(t) \sin xt \, dt \\
= \int_0^{\pi/2x} \left[ a(t) - a\left(\frac{2\pi}{x} - t\right) \right] \sin xt \, dt \\
\geq Ax \int_0^{\pi/2x} ta(t) \, dt.
\]

Thus we get (8), and hence the theorem is proved.

3. Let us now prove Theorem 4. Let
\[
f_1(x) = \int_0^\infty a(t) \cos xt \, dt
= -\frac{1}{x} \int_0^\infty a'(t) \sin xt \, dt = -\frac{1}{x} \int_0^{\pi/2x} b'(t) \sin xt \, dt,
\]
where \( b'(t) \) denotes the term-wise differentiated series of \( b_n(t) \) by \( t \).

Hence, from the proof of Theorem 3, we get the required result.

Finally we shall prove the following

**Theorem 6.** Let \( a(x) \) be a positive, decreasing and convex function in the interval \((0, \infty)\), tending to zero as \( x \to \infty \). Let \( a_n = a(n) \) and \( \sum a_n = \infty \). Then
\[
f(x) \leq A \int_0^{1/x} t |a'(t)| \, dt \quad \text{as} \quad x \to 0.
\]

For, we write
\[
f_1(x) = \int_0^\infty a(t) \cos xt \, dt = \int_0^{\pi/2x} c_\alpha(t) \cos xt \, dt,
\]
where
\[
c_\alpha(t) = \sum_{k=1}^\infty (-1)^k \left[ a\left(\frac{k\pi}{x} + t\right) - a\left(\frac{(k+1)\pi}{x} - t\right) \right].
\]

By the convexity of \( a(t) \),
\[
cia(t) \leq a(t) - a(\pi/x - t),
\]
and then
\[
f_1(x) = \int_0^{\pi/2x} \cos xt \left[ a(t) - a(\pi/x - t) \right] \, dt
= -\int_0^{\pi/2x} \cos xt \, dt \int_0^{\pi/x - t} a'(u) \, du
= -\int_0^{\pi/2x} a'(u) \, du \int_0^{\pi/2x} \cos xt \, dt - \int_0^{\pi/2x} a'(u) \, du \int_0^{\pi/x - u} \cos xt \, dt
\leq -A \int_0^{\pi/2x} a'(u) \sin xu \, du \leq -A \int_0^{\pi/2x} a'(u) \, du.
\]

Thus we get the required inequality.

**References**