80. A Note on the Motion of Saturn's Satellites—Enceladus and Dione

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In the present paper I propose to develop a new theory of the motion of Enceladus and Dione by taking terms of higher orders and of higher degrees into account and to redetermine the masses of these two satellites and the constants of the figure of Saturn.

1. The equations of motion and the solutions.1) As is well known the mean motions of Enceladus and Dione are nearly commensurable with each other. Hence, in order to get the expressions of the secular motions of the apses as far as the terms of the second order, we must integrate the following system of the simultaneous linear differential equations written in matrix notation:

\[
\frac{d}{dt} \mathbf{r} = \mathbf{u} \mathbf{r} + \mathbf{c} = (\mathbf{a} + \mathbf{b}_1 + \mathbf{b}_2) \mathbf{r} + \mathbf{c},
\]

where the matrix \( \mathbf{r} \) is unknown and the matrices \( \mathbf{u} \) and \( \mathbf{c} \) are known functions of \( t \). \( \mathbf{u} \) is divided into three parts; \( \mathbf{a} \) is constant, \( \mathbf{b}_1 \) depends on the ellipticity of Saturn, and \( \mathbf{b}_2 \), as well as \( \mathbf{c} \), is derived from the disturbing function due to the attraction of the other satellite.

We denote Delaunay's canonical orbital elements of Enceladus by \( \Lambda, \lambda, \xi, \eta, \psi, p, \) and \( q \), those of Dione by the corresponding letters with suffix 1, and the mean value of \( \Lambda \) by \( \Lambda^0 \). Put

\[
\begin{align*}
\xi &= \frac{\xi}{a^2} + \frac{k}{a^2} \cos \lambda, \\
\eta &= \frac{\eta}{a^2} - \frac{k}{a^2} \sin \lambda, \\
\Lambda &= \Lambda^0 + \delta \Lambda, \\
\lambda &= \kappa t + \varepsilon + \delta \lambda,
\end{align*}
\]

where \( k \) represents the constant of the figure of Saturn, and\(^2\)

\[
\begin{align*}
2(1) &= ma(50)\,(2) = ma(70)^{(1)}, \\
2(1) &= ma(50)^{(2)}, \\
2(2) &= ma(70)^{(3)}.
\end{align*}
\]

Then it can be shown that the explicit expressions of \( r, b_2, \) and \( c \) are of the form:

1) The full and detailed text of this part of the present paper will be published in a coming number of the Publ. Astr. Soc. Japan.
\[
\begin{pmatrix}
\frac{\delta A}{A^0} \\
\frac{\delta A_1}{A_1^0} \\
\xi_1 \\
\xi_2 \\
\delta \lambda \\
\gamma \\
\eta \\
\eta_1
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
-(1)n \sin (2\lambda_1 - \lambda) \\
-(2)n_1 \sin (2\lambda_1 - \lambda) \\
0 \\
0 \\
-(1)n \cos (2\lambda_1 - \lambda) \\
-(2)n_1 \cos (2\lambda_1 - \lambda)
\end{pmatrix},
\]

\[b = \begin{pmatrix}
0 & S_1 & 0 & C_1 \\
0 & 0 & -C_2 & 0 \\
0 & 0 & 0 & O \\
0 & 0 & S_2 & 0
\end{pmatrix},\]

where

\[S_1 = \begin{pmatrix}
(1)n \sin (2\lambda_1 - \lambda) \\
-2(1)n_1 \sin (2\lambda_1 - \lambda) \\
-(1)n \sin (2\lambda_1 - \lambda) \\
-2(2)n_1 \sin (2\lambda_1 - \lambda)
\end{pmatrix},
\]

\[S_2 = \begin{pmatrix}
(2)n \sin (2\lambda_1 - \lambda) \\
-2(2)n_1 \sin (2\lambda_1 - \lambda) \\
2(1)n \sin (2\lambda_1 - \lambda) \\
2(2)n_1 \sin (2\lambda_1 - \lambda)
\end{pmatrix}.
\]

By substituting \textquoteleft\textquoteleft cos\textquoteright\textquoteright\ for \textquoteleft\textquoteleft sin\textquoteright\ in these expressions of \(S_1\) and \(S_2\) we get those of \(C_1\) and \(C_2\) respectively.

The effect of the neglected terms depending on \(p\) and \(q\) is taken into account by adding correction terms to the elements of the matrix \(a\).

By the method of H. F. Baker\(^3\) for integrating linear differential equations, we get the solution of (1) written in the following form:

\[(4) \quad \xi = \Omega(u)t + \Omega(u)Q[\Omega^{-1}(u)c].\]

Picking out the terms containing \(t\) as the explicit linear factor in the matrix series \(\Omega(u)\) and putting \(t=0\) in these coefficients of \(t\), we have the matrix from which is formed the equation to be satisfied by the characteristic exponents. Then by solving the characteristic equation thus derived we obtain the following expressions for the secular motions \(b\) and \(b_1\) of the apses:

\[(5) \quad \begin{aligned}
\frac{b}{n} &= \frac{k}{a^2} + \frac{m_1 a(2)^0}{2} + \frac{5}{2} \frac{l}{a^4} \frac{1}{a^2} \left( \frac{k}{a^2} \right)^2 \\
&+ \frac{\nu}{n} \left\{ \exp \left[ \frac{3}{2} \left( \frac{n}{\nu} \right)^2 + 4(1) \left( \frac{n_1}{\nu_1} \right)^2 \right] \left( \frac{1}{\nu} \left( \frac{n}{\nu} \right)^2 \right) - 1 \right\}, \\
\frac{b_1}{n_1} &= \frac{k}{a_1^2} + \frac{m_1 a(2)^0}{2} + \frac{5}{2} \frac{l}{a_1^4} \frac{1}{a_1^2} \left( \frac{k}{a_1^2} \right)^2 \\
&+ \frac{\nu_1}{n_1} \left\{ \exp \left[ \frac{3}{2} \left( \frac{n}{\nu_1} \right)^2 + 4(2) \left( \frac{n_1}{\nu_1} \right)^2 \right] \left( \frac{1}{\nu_1} \left( \frac{n}{\nu} \right)^2 \right) - 1 \right\},
\end{aligned}\]

where

\[ \nu = 2n_1 - n, \quad \nu_1 = 2n_1 - n - b. \]
The secular motion \( h \) of the node of Enceladus obtained by a similar method is expressed as,

\[
\frac{h}{n} = -\frac{k}{a^2} + \frac{m_1 a (11)^{10}}{2} - \frac{5}{2} \left( \frac{k}{a^2} \right) + 3 \left( \frac{k}{a^2} \right)^2.
\]

The formulae of H. Struve\(^4\) and others\(^5\) for the secular motions consist of the first three terms on the second-hand sides of (5) and (6). The fourth terms which D. Brouwer\(^7\) has already found are also derived from the author’s theory of the secular perturbations of asteroids.\(^8\) The fifth terms in (5), which are of higher orders from the formal point of view, take, especially for Enceladus, the sensible values, \((3°.37/n\) for Enceladus and \(0°.004/n_1\) for Dione), because the values of \(\nu\) and \(\nu_1\) are sufficiently small.

The solution (4) shows us that the forced eccentricity \( f \) and the periodic terms of the mean longitudes are expressed as:

\[
f = -(1) \frac{n}{\nu},
\]

\[
\begin{align*}
\delta \lambda &= r \sin (\mu + \nu t) + s \sin (\nu_1 + \nu t), \\
\delta \lambda_1 &= -\frac{1}{2} \frac{m}{m_1} \frac{a}{a} \delta \lambda,
\end{align*}
\]

where

\[
r = -3(1)c \left( \frac{n}{\nu} \right)^2, \quad s = 3(2)c_1 \left( \frac{n}{\nu_1} \right)^2.
\]

c and \(c_1\) represent the respective proper eccentricities and \(\mu\) and \(\mu_1\) are arbitrary constants.

J. Woltjer\(^9\) has pointed out that \(\nu\) in \(\delta \lambda\) must be replaced by \(\tau\) which is \(\nu\) plus term analogous to the fifth of the upper formula of (5). We are inclined to think that no essential discrepancy exists between Woltjer’s and the present theory, but the only and the important difference is that \(\nu\) appearing in the denominator of the present formulae is the same as Woltjer’s \(\tau\).

2. The secular motions for Mimas and Tethys.\(^1\) The ratio of the mean motions of Mimas and Tethys is also nearly 2, but \(\nu\) and \(\nu_1\) are not so small as in the previous case. On the other hand the argument \(4\lambda - 2\lambda - \theta - \theta_1\) is of libration for this system. And the values of the eccentricities and of the inclinations except for the eccentricity of Tethys are rather large, so the terms of higher

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9) J. Woltjer: B. A. N., 5 (1922); ibid., 7 (1922).
degrees contribute by an appreciable amount to the secular motions. The corresponding sensible terms for Mimas are:

\[
\begin{align*}
\frac{b}{n} &= k \left( \frac{1}{2} A^2 - 2B^2 \right) \frac{5}{2} a^2 + m_1 a^{(2)^{10}} \\
\frac{h}{n} &= -k \left( \frac{3}{2} A^2 + 2B^2 - \frac{1}{2} \right) \frac{5}{2} a^2 + m_1 a^{(11)^{10}}
\end{align*}
\]

where \( A \) and \( B \) represent respectively the values of the proper eccentricity and inclination.

3. The masses of Enceladus and Dione. G. Struve\(^5\) has reduced the observational data up to 1928 and gave the following numerical result.

\[
\begin{align*}
2n_1 - n &= 123^\circ.43, \quad \text{(per year)}, \\
\nu &= 32^\circ.51 \pm 0^\circ.20, \quad \nu_1 = 98^\circ.14 \pm 0^\circ.27, \\
\rho &= 14^\circ.39 \pm 0^\circ.81, \quad r_1 = 0^\circ.93 \pm 0^\circ.24, \\
s &= 14^\circ.06 \pm 0^\circ.80, \quad s_1 = 0^\circ.91 \pm 0^\circ.24, \\
e = f &= 0.00448 \pm 0.00014, \quad e_1 = c_1 = 0.00221 \pm 0.00006.
\end{align*}
\]

The mass of Dione is derived from the value of the forced eccentricity of Enceladus by using the formula (7), in which \( \nu \) is substituted by the observed value 32\(^\circ\).51. The result is

\[
m_1 = (2.015 \pm 0.076) \times 10^{-6},
\]

expressed as the fraction of the mass of Saturn.

On the other hand H. Struve\(^6\), G. Struve\(^5\), and Jeffreys\(^6\) adopted 29\(^\circ\).3 as the value of \( \nu \), which they computed from the secular motion calculated by H. Struve’s formula. So it is evident that their value of the mass of Dione is different from the mass derived in the present paper even though the same data are used in both calculations.

From the coefficient \( s \) in the expression \( \delta \lambda \) the mass of Dione is also calculated to be

\[
m_1 = (2.12 \pm 0.12) \times 10^{-6}.
\]

By taking the weighted mean of these two values we get as its definite mass,

\[
m_1 = (2.045 \pm 0.064) \times 10^{-6}.
\]

The ratio of the masses of Enceladus and Dione, which is related to that of the coefficients of the periodic parts of the longitudes, is easily computed to be 0.082 \pm 0.021, so the value of the mass of Enceladus is found to be

\[
m = (1.68 \pm 0.48) \times 10^{-7}.
\]

4. The constants of the figure of Saturn. We adopt as the masses of Enceladus and Dione the values determined in the previous section, and as those of the other satellites the values computed by
Jeffreys. By substituting for the Struve's constants of the figure of Saturn the values,
\[ \frac{k}{a_0^2} = 0.024305, \quad \frac{l}{a_0^4} = 0, \]
we calculate by the formulae (5), (6), and (9) the respective values of the secular motions. The result is compared with the observed value in Table I.

Table I

<table>
<thead>
<tr>
<th></th>
<th>Mimas</th>
<th>Enceladus</th>
<th>Tethys</th>
<th>Dione</th>
<th>Rhea</th>
<th>Titan</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Apse</td>
<td>Node</td>
<td>Apse</td>
<td>Node</td>
<td>Apse</td>
<td>Node</td>
</tr>
<tr>
<td>Obs.</td>
<td>365°.60</td>
<td>365°.23</td>
<td>155°.89</td>
<td>72°.23</td>
<td>30°.75</td>
<td>10°.20</td>
</tr>
<tr>
<td>P.e.</td>
<td>±0.07</td>
<td>±0.07</td>
<td>±0.20</td>
<td>±0.02</td>
<td>±0.16</td>
<td>±0.03</td>
</tr>
<tr>
<td>Cal.</td>
<td>365°.36</td>
<td>359°.82</td>
<td>154°.59</td>
<td>71°.85</td>
<td>30°.49</td>
<td>10°.07</td>
</tr>
<tr>
<td></td>
<td>365°.71</td>
<td>365°.17</td>
<td>155°.89</td>
<td>72°.23</td>
<td>30°.58</td>
<td>10°.07</td>
</tr>
</tbody>
</table>

In Table I we have taken \( b = \nu + (2n_1 - n) \) as the observed value for the apse of Enceladus, and for Titan the solar action is already subtracted from the weighted mean of the observed motions of the apse and the node. These two values are known from the theoretical standpoint not to be sensibly different from each other except for their signs.

Now by the method of least squares, after giving the weight inversely proportional to the square of the observed probable error, we get the corrected values of \( k \) and \( l \) as follows:

\[
\begin{align*}
&\frac{k}{a_0^2} = 0.024280 \pm 0.000010, \\
&\frac{l}{a_0^4} = (1.45 \pm 0.04) \times 10^{-3}.
\end{align*}
\]

The calculated secular motions with these new values are given on the last line of Table I.