54. Evans-Selberg's Theorem on Abstract Riemann Surfaces with Positive Boundaries. II

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Our $N_{\nu,\mu}(p, q)$ is increasing with respect to $m$. We define the value of $N(z, q)$ at a minimal point $p$ by $\lim_{m \to M'} N_{\nu,\mu}(p, q)$ denoted by $N(p, q)$. If $p$ or $q$ belongs to $R$, this definition is equivalent to that defined before.

If $V_m(p)$ is not regular, we define $N_{\nu,\mu}(p, q)$ by $\lim_{m' \to m} N_{\nu,\mu}(p, q)$, where $m' < m$ and $V_{m'}(p)$ is regular. In the case when $V_m(p)$ is regular, it is proved that $\lim_{m' \to m} N_{\nu,\mu}(p, q) = N_{\nu,\mu}(p, q)$, hence we can define $N_{\nu,\mu}(p, q)$ for every $m < \sup_{z \in R} N(z, p) = M'$. As in case of a Riemann surface with a null-boundary, we can prove the following

Theorem 10. 1) $N(z, q)$ $(q \in \overline{R})$ is $\delta$-lower semicontinuous in $R + B_1$.
2) $N(z, q)$ is superharmonic in weak sense at every point of $R + B_1$.
3) If $p$ and $q$ are in $R + B_1$, then $N(p, q) = N(q, p)$.

Till now $N(z, q)$ $(q \in \overline{R})$ is defined only on $R + B_1$. Next we define $N(z, q)$ at points belonging to $B_0$. If $p \in B_0$, $N(p, q) = \int_{p} N(z, p_a) d\mu(p_a)$ $(p_a \in B_1)$ by Theorem 8. Although the uniqueness of this mass distribution is not proved by the present author, the value of $N(z, q)$ in $R + B_1$ is uniquely determined. On the other hand, by 3), for $q \in B_1$, $N(p, q) = N(q, p)$. Hence it is quite natural to define the value of $N(z, q)$ at $p \in B_0$ by $\int N(p_a, q) d\mu(p_a)$. Evidently by 3), in such definition, we have $N(q, p) = N(p, q)$, where the term of the right hand side does not depend on a particular distribution but on the behaviour of $N(z, q)$, because $N(p, q) = \lim_{m \to M'} N_{\nu,\mu}(p, q)$ and $N_{\nu,\mu}(p, q)$ is defined by the value of $N(z, q)$ on $\partial V_m(p)$.

Theorem 11. 1) If $q \in R + B_1$, then $N(p, q) = N(q, p)$ for $p \in \overline{R}$.
2) If $q \in \overline{R}$ and $p \in R + B_1$, then $N(p, q) = \int N(p, q_a) d\mu(q_a)$, where $N(z, q_a) = \int N(z, q) d\mu(q_a)$.
3) $N(z, q)$ $(q \in \overline{R})$ is $\delta$-lower semicontinuous in $\overline{R}$.
1') For every \( p \) and \( q \) belonging to \( \overline{R} \), \( N(p, q) = N(q, p) \).

7. Potentials on \( \overline{R} \). In the sequel, we shall study the mass distributions on \( \overline{R} \). We have seen that \( N(z, p) \) has the essential properties of logarithmic potential: lower semicontinuity in \( \overline{R} \), symmetricity and superharmonicity in \( \overline{R} + B_1 \). But there exists the fatal difference between our case and space, that is, the real mass distribution can be defined only on \( R + B_1 \), i.e. the distribution on \( B_0 \) is superficial and it can be replaced, by Theorem 8, by that on \( B_1 \) where \( N(z, p) \) is superharmonic. Therefore only subsets \( R + B_1 \) of \( \overline{R} \) can be a kernel of mass distribution. Hence it is easy to construct the potential theory on \( \overline{R} \).

The energy integral \( I(\mu) \) of a mass distribution \( \mu \) on a \( \delta \)-closed subset of \( R + B_1 \) defined as in space

\[
I(\mu) = \int \int N(q, p) d\mu(p) d\mu(q)
\]

and the capacity is defined as usual. In § 1, we defined capacity of \( F \), we must study the relation between two capacities. At first, we have, if \( \text{Cap}(F) > 0 \), \( \text{Cap}^*(F) > 0 \). Now we have the following

**Theorem 12.** Let \( F \) be a \( \delta \)-closed subset of \( R + B_1 \) of capacity positive. Then there exists a unit mass distribution on \( F \) whose energy is minimal and whose potential \( U(z) \) has the following properties:

1) \( U(z) \) is a constant \( C \) on the kernel of this distribution.

2) \( U(z) = C \) on \( E \) except possibly a set of capacity zero.

3) \( U(z) = U_F(z) \).

4) \( U(z) = C_\omega_F(z) \), where \( \omega_F(z) \) is the equilibrium potential of \( F \).

By 2) of this theorem and by 2) of Theorem 5, we have the following

**Corollary.** \( \text{Cap}(F) = \text{Cap}^*(F) \).

**Transfinite diameter.** Since \( N(z, p) \) \( (p \in \overline{R}) \) is \( \delta \)-lower semicontinuous in \( \overline{R} \), the transfinite diameter of a \( \delta \)-closed subset \( A \) of \( \overline{R} \) is defined as follows:

\[
\frac{1}{D_A} = \lim_{n \to \infty} \left( \min \left( \frac{1}{n} \sum_{i,j=1}^{n} (p_i, p_j) \right) \right).
\]

Then as in the case of \( R^* \) with a null-boundary, we have the following

**Theorem 13.** If \( D_A = 0 \) for a \( \delta \)-closed subset \( A \) of \( \overline{R} \), then there exists a superharmonic function \( U(z) \) in \( \overline{R} \) such that \( U(z) = 0 \) on \( \partial R_0 \), \( \int A U(z) \, ds = 2\pi \) and \( U(z) = \infty \) at every point of \( A \).

For a \( \delta \)-closed subset \( A \) of \( R + B_1 \), it can be proved as in space.
$D_A = \frac{1}{I(\mu)}$, where $I(\mu)$ is the energy of the equilibrium potential of $A$. Hence we have the following

**Theorem 14 (Extension of Evans-Selberg’s theorem).** Let $A$ be a $\delta$-closed subset of $R+B_1$, of capacity zero. Then there exists a unit mass distribution on $A$ whose potential satisfies the following properties:

1) $U(z) = 0$ on $\partial R_0$.
2) $U(z) = \infty$ at every point of $A$.
3) $U(z) = U_A(z)$.
4) $\int_{\partial R_0} \frac{\partial U(z)}{\partial n} ds = \int_{C_r} \frac{\partial U(z)}{\partial n} ds$, for the niveau curve $C_r$ of $U(z)$ with $r \in E$, where $E$ is a set in the interval $[0, \infty]$ such that $\text{mes} E = 0$.

In general cases we can not omit the condition that $A$ is a subset of $R+B_1$. The reason is as follows: there may exist a set $B_0$ which is an $F_\sigma$ and of capacity zero and any mass can not be distributed on $B_0$, in other words, $B_0$ has behaviour like an empty set in space for mass distribution though $B_0$ is not empty.

The value of $U(z)$ at a point $p \in B_0$ is given as follows: since $N(z, p) = \int_{n_1} N(z, p_1) d\mu(p_1)$ ($p_1 \in B_1$), $U(p) = \int U(p_1) d\mu(p_1)$. Therefore $U(z)$ may be infinite at larger set $A'$ than $A$. 