75. Notes on Topological Spaces. III. On Space of Maximal Ideals of Semiring

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Definition 1. A semiring $A$ is an algebra with two binary operations, addition (written $+$) which is associative, and multiplication which is associative, and satisfies the distributive law

$$a(b+c) = ab + ac, \quad (b+c)a = ba + ca.$$ 

In this paper, we suppose that $A$ has the further properties:

1) There are two elements $0, 1$ such that

$$x + 0 = x, \quad x \cdot 1 = x$$

for every $x$ of $A$.

2) Two operations, addition and multiplication, are commutative.

Definition 2. A non-empty proper subset $I$ of $A$ is called an ideal, if

1) $a, b \in I$ implies $a + b \in I$,
2) $a \in I, x \in A$ implies $ax \in I$.

W. Slowikowski and W. Zawadowski [6] proved that every ideal is contained in a maximal ideal. An ideal is maximal if there is no ideal containing properly it.

Let $\mathcal{M}$ be the set of all maximal ideals in a semiring $A$. We shall define two topologies on $\mathcal{M}$.

For every $x$ of $A$, we denote by $\mathcal{J}_x$ the set of all maximal ideals containing $x$, and by $I_x$ the set $\mathcal{M} - \mathcal{J}_x$, i.e. the set of all maximal ideals not containing $x$. Let $I$ be an ideal of $A$, we denote by $\mathcal{J}_I$ the set of all maximal ideals containing $I$.

We shall choose the family $\{\mathcal{J}_x | x \in A\}$ as a subbase for open sets of $\mathcal{M}$. We shall refer to the resulting topology on $\mathcal{M}$ as $\mathcal{J}$-topology (in symbol, $\mathcal{M}_J$). Similarly, we shall take the family $\{I_x | x \in A\}$ as a subbase for open sets of $\mathcal{M}$ (in symbol, $\mathcal{M}_I$). These two topologies for normed ring or general commutative ring were considered by I. Gelfand and G. Silov [2] or P. Samuel [5].

Let $M_1, M_2$ be two distinct elements of $\mathcal{M}$. Then we have $M_1 + M_2 = A$. Therefore there are $a, b$ such that $a + b = 1$ and $a \in M_1, b \in M_2$. 

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Let $b \in M_2$, so we have $\mathcal{J}_a \ni M_1$, $\mathcal{J}_b \ni M_2$ and $\mathcal{J}_a \cap M_b = 0$. Hence

Theorem 1. The topological space $\mathfrak{M}_A$ is a $T_2$-space.

Let $M$ be an element of $\mathfrak{M}_F$, and $M \ni M_1 \in \mathfrak{M}_F$, then there is an element $a$ such that $a \in M_1$ and $a \notin M$. Therefore $\mathcal{J}_a \ni M_1$ and $\bigcap_{x \in M} \mathcal{J}_x \ni M_1$. This implies $M = \bigcap_{x \in M} \mathcal{J}_x$. Hence we have the following

Theorem 2. The topological space $\mathfrak{M}_F$ is a $T_1$-space.

Let $I$ be an ideal of $A$ and $\{a_\lambda\}$ a generator of $I$, then we have

$$\mathcal{J}_I = \bigcap_{\lambda} \mathcal{J}_{a_\lambda}.$$ 

Therefore, the closed sets for the topological space $\mathfrak{M}_F$ have the form $\mathcal{J}_{I_1} \cap \mathcal{J}_{I_2} \cap \cdots \cap \mathcal{J}_{I_n}$, where $I_i$ are ideals of $A$.

Let $I = \bigcap_{i=1}^n I_i$, if $\mathcal{J}_{I_i} \ni M$ for some $i$, then $M \ni I_i$ and $M \ni I$. This implies $\mathcal{J}_I \ni M$ and we have $\bigcap_{i=1}^n \mathcal{J}_{I_i} \subset \mathcal{J}_I$. Suppose that there is a maximal ideal $M$ such that $M \ni \bigcap_{i=1}^n \mathcal{J}_{I_i}$, then $M \ni \mathcal{J}_I$ and $M \ni \bigcup_{i=1}^n \mathcal{J}_{I_i}$. Hence $M \ni I$ and $M$ does not contain every $I_i$ ($i = 1, 2, \cdots, n$). Therefore, since $M$ is a maximal ideal, there are elements $a_i \in I_i$ and $m_i \in M$ such that

$$a_i + m_i = 1 \quad (i = 1, 2, \cdots, n).$$

Thus, we have

$$1 = a_1 a_2 \cdots a_n + m, \quad m \in M$$

and $a_1 a_2 \cdots a_n \in I$. This implies $I + M = A$. Hence, by $I \subset M$, we have $M = A$, which is a contradiction. This shows the following relation:

$$\bigcup_{i=1}^n J_{I_i} = \mathcal{J}_I$$

and we have the following

Theorem 3. The closed sets for $\mathfrak{M}_F$ are expressed by sets $\mathcal{J}_I$, where $I$ is an ideal of $A$.

By Theorem 3, we shall show the following

Theorem 4. The space $\mathfrak{M}_F$ is a compact $T_1$-space.

To prove it, let $\{J_{I_\lambda}\}$ be a family of closed sets in $\mathfrak{M}_F$ with the finite intersection property, where $I_\lambda$ are ideals in $A$. Therefore, any finite family of $I_\lambda$ does not generate the semiring $A$. Hence the ideal $I$ generated by $\{I_\lambda\}$ does not contain the unit 1 of $A$. This shows that $I$ is contained in a maximal ideal $M$. Hence

$$\bigcap_{\lambda} J_{I_\lambda} \ni M.$$ 

Therefore, since $\bigcap_{\lambda} J_{I_\lambda}$ is non-empty, $\mathfrak{M}_F$ is a compact space.

Example. Let $A$ be the semiring of non-negative integers with ordinary addition and multiplication. An ideal $I$ of $A$ is maximal, if and only if, there is a prime number $p$ such that $I = \langle p \rangle$. As closed set of $\mathfrak{M}_F$ is finite, any distinct two elements of $\mathfrak{M}_F$ can not separate
by disjoint open sets. Hence $\mathcal{M}$ for $\Gamma$-topology is not a $T_\sigma$-space.

Following W. Slowikowski and W. Zawadowski [6], we shall define positive semirings.

Definition 3. A semiring $A$ is positive, if, for every $a$ of $A$, $1 + a$ has an inverse.

Let $A$ be a positive semiring, then, for any element $a$ of $A$, there is an element $b$ such that $ab + b = 1$, i.e. $(a) + (b) = A$. This means that, for every element $a$ of a positive semiring $A$, $A$ contains at least one element $b$ such that $A$ is generated by $a$ and $b$. Hence any maximal ideal $M$ containing $b$ does not contain $a$. Consequently $\Delta_b \subset \Gamma_a$. Hence we have

Lemma 1. Every open set of $\mathcal{M}_r$ for a positive semiring contains an open set of $\mathcal{M}_s$.

Any set $\Gamma_a$ is a closed set for $\mathcal{M}_s$. If $\Gamma_a$ is a closed set for $\mathcal{M}_r$, then there is an ideal $I$ of $A$ such that $\Gamma_a = A$ by Theorem 3. If $(a) + I = A$, then there is a maximal ideal $M$ containing $(a) + I$, and $\Gamma_a \neq M$ and $M \in J$, therefore this implies $\Gamma_a \neq M$. Hence we have $(a) + I = A$, so there are such elements $x \in A$ and $b \in I$ that $ax + b = 1$.

This shows that any maximal ideal containing $b$ does not contain $a$. Hence $\Delta_b \subset \Gamma_a$. Clearly, $\Delta_b \subset J_b$. Therefore $\Delta_b = \Gamma_a$ by $\Gamma_a = A$.

Lemma 2. If $\Gamma_a$ is closed for $\mathcal{M}_r$ of a positive semiring, then there is an element $b$ such that $\Delta_b = \Gamma_a$.

Conversely, we have easily the following

Lemma 3. If, for any element $a$ of $A$, there is an element $b$ such that $\Gamma_a = A_b$, then $\Gamma$-topology and $\Delta$-topology on $\mathcal{M}$ coincide.

Hence we have the following

Theorem 5. $\Gamma$-topology and $\Delta$-topology for $\mathcal{M}$ of a positive semiring $A$ coincide, if and only if, for every $a$ of $A$, there is an element $b$ of $A$ such that maximal ideals not containing $a$ are same of the family of maximal ideals containing $b$.

Definition 4. If for every two maximal ideals $M, N$ in a semiring $A$, there are two elements $x \in M, y \in N$ such that $xy$ is contained in the intersection of all maximal ideals of $A$, $A$ is called normal.

It is known that $A$ is normal, if and only if $\mathcal{M}$ is a normal space (see W. Slowikowski and W. Zawadowski [6]).

Therefore we have

Theorem 6. If, for any element $a$ of $A$, there is an element $b$ such that $\Gamma_a = A_b$, then $A$ is normal.

Theorem 7. If any $\Gamma_a$ is closed of $\mathcal{M}_r$ of a positive semiring $A$, then $A$ is normal.

In our later paper, we shall investigate the ideal structure of semiring and general theory of topological semiring.
References


