1. Let $L$ be a complete lattice-ordered semigroup (cl-semigroup) with a maximally integral identity $e$, and suppose that $L$ has a unique mapping into itself $a \rightarrow a^{-1}$ with two properties 1) $aa^{-1}a \leq a$ and 2) $axa \leq a$ implies $x \leq a^{-1}$. In the previous paper [1], we obtained that $L$ forms a commutative cl-group which is a direct product of infinite cyclic groups generated by prime elements, if $L$ satisfies the following conditions:

(1) The ascending chain condition (a.c.c.) holds for integral elements of $L$.

(2) Any prime element is divisor-free (maximal).

(3) Any prime element contains an element $c$ satisfying $c^{-1}c = c$.

Our purpose of the present note is to show that the condition (1) is replaceable equivalently by the restricted descending chain condition for integral elements of $L$.

2. Let $L$ be a cl-semigroup with an identity $e$. If $e$ is maximally integral, then, in order that $L$ has a mapping into itself $a \rightarrow a^{-1}$ with above two properties 1) and 2), it is necessary and sufficient that $L$ forms a residuated lattice. In [1] we have proved that the condition is necessary. We show that the condition is sufficient. Suppose that $L$ is a residuated lattice. Then $(e:a)_{l} = (e:a)_{r}$. For, let $ax \leq e$, then $xaxa \leq xa$, $(xa^{-1}e)^{2} \leq xa \leq e$, and so $xa \leq e$, $xa \leq e$. Hence $(e:a)_{l} \leq (e:a)_{r}$. Similarly $(e:a)_{r} \leq (e:a)_{l}$. We get therefore $(e:a)_{l} = (e:a)_{r}$. We next prove that $e = (a:a)_{l} = (a:a)_{r}$. Since $(a:a)_{l}a \leq a$, we have $(a:a)_{l}a \leq (a:a)_{r}$. $(a:a)_{r} \geq e$ is evident. Hence $e = (a:a)_{l}$, similarly $e = (a:a)_{r}$. We now define a mapping $a \rightarrow a^{-1}$ with $a^{-1} = (e:a)_{l} = (e:a)_{r}$. Then $aa^{-1}a = a \cdot (e:a)_{l}a \leq ae = a$, and $axa \leq a$ implies $ax \leq (a:a)_{l} = e$, hence $x \leq (e:a)_{l} = a^{-1}$.

Lemma 1. Let $a$ and $b$ be two elements in $L$. If $b$ covers $a$, then $(a:b)$ is a prime element. In particular, if $b$ is integral, then $(a:b)$ is a prime element containing $a$. Similarly for $(a:b)_{r}$.

Proof. Suppose that $bx \leq a$. Then $abx \leq a^{2} \leq ab$. Hence $x \leq (ab:ab)$.

1) An element $x$ is called integral if $x^{2} \leq x$. $e$ is called maximally integral if $e^{2} \leq e$.

2) Cf. [1, p. 14, Theorem 2.6].

3) Cf. [2, p. 201]. $(a:b)$ will denote the left residual of $a$ by $b$ which is the largest $x$ satisfying $bx \leq a$. Symmetrically for the right residual $(a:b)_{r}$ of $a$ by $b$.

4) Cf. [1, p. 12, Theorem 2.2].
Let $u$ and $v$ be two integral elements such that $uv < (a : b)$ and $u < (a : b)$. Then $bu = a$ and $bu < a$. Hence $a < bu$, $a < bu$, $a < bu$. This implies $b = bu$, and so $bv = buv < av < a$, $v < (a : b)$. This shows that $(a : b)$ is prime. Similarly $(a : b)_r$ is prime. The other part of this lemma is evident.

In the following we suppose that any prime is divisor-free.

**Lemma 2.** Let $a$ be an integral element of $L$, and $X$ a set of elements $x$ such that $x^a < a$ for a suitable whole number $a = \sigma(x)$. If the descending chain condition (d.c.c.) holds for the interval $e[a] = [y; a \leq y \leq e]$, then there exists a whole number $\rho$ such that $(\sup X)^\rho < a$.

**Proof.** If the set $X$ consists of the element $a$ only, then our assertion is trivial. We assume that $X$ contains at least two elements. Then evidently $u = \sup X > a$. We find now that $u$ is not an idempotent. For, let $u$ be an idempotent. Since $eu > u^2 = e$, we have $(a : u) = e$. Take an element $m$ which covers $(a : u)$. Then $p = (a : u, m)$ is a prime element, and so $p$ is divisor-free. If $e = u \bowtie p$, then $e = (\sup X) \bowtie p = \sup_{x \in X} (x \bowtie p)$. Hence there exists $x_0 \bowtie p$ ($x_0 \in X$) such that $e = x_0 \bowtie p$.

Since there exists a whole number $\sigma$ such that $x_0^\sigma < a$, we obtain $e = e^\sigma = (x_0 \bowtie p)^\sigma = \bigcup_j f_j p g_j \leq p$, a contradiction. Now, if $u \bowtie p = p$, then $u \leq p$. On the other hand, since $mpu < a$, we obtain $mu = mw^2 < mpu < a$. Hence $m \leq (a : u)_r$. This is a contradiction. Repeating the above arguments to the set $X^u$, we obtain $u^2 = (\sup X)u = \sup (X^u) > (\sup (X^u))^2 = u^4$. Continuing in this way we have $u > w^2 > u^4 > \cdots$, $w < a$.

**Lemma 3.** Let $a$ be an integral element of $L$. If the d.c.c. holds for the interval $e[a]$, then $a$ contains a product of finite number of primes.

**Proof.** Let $X$ be the set of all elements $x$ such that $x^a < a$ for a suitable whole number $a$. Take an element $c_1$ which covers $u = \sup X$. Then $p_1 = (u : c_1) = e$ and $p_1$ is a prime element. If $c_1 \leq p_1$, then $c_1 \leq p_1 c_1 \leq u$, $c_1 \in X$, a contradiction. Hence $c_1 \leq p_1$. If $p_1 \parallel u$, then we take an element $c_2$ such that $c_2 \leq p_1$ and $c_2$ covers $u$. Put $p_2 = (u : c_2)$. Then, since $c_2 \leq p_2$ and $c_2 \leq p_1$, the prime element $p_2$ ($\parallel u$) is not equal to $p_1$. If $p_1 \bowtie p_2 \parallel u$, then we take an element $c_3$ such that $c_3 \leq p_1 \bowtie p_2$ and $c_3$ covers $u$. Put $p_3 = (u : c_3)$. Then $p_3$ ($\parallel u$) is not equal to $p_1$ and $p_2$. Continuing in this way, we obtain, after a finite number of steps, $p_1 \bowtie \cdots \bowtie p_n = u$. Since there exists a whole number $\rho$ such that $w^\rho < a$, we obtain

$$(p_1 \bowtie \cdots \bowtie p_n)^\rho \leq (p_1 \bowtie \cdots \bowtie p_n)^\rho = w^\rho < a.$$ 

This proves our assertion.

**Lemma 4.** Suppose that the restricted descending chain condition

5) If $x^a < a$ ($x \in X$), then $(xu)^a < a$.
6) If $p_1 = e$, then $c_1 = ec_1 \leq p_1 c_1 \leq u$, a contradiction.
(r.d.c.c.) holds for integral elements in L, and any prime contains an element c satisfying \(c^{-1} = c\). If both a and \(a^{-1}\) are integral, then \(a = e\).

**Proof.** Let \(a \leq e, a \neq e\). Using Lemma 1, we can take a prime element \(p\) such that \(a \leq p < e\). Since \(e \geq a^{-1} \geq p^{-1} \geq e^{-1} = e\), it follows that \(a^{-1} = p^{-1} = e\). Let \(c = c^{-1}\) be an element contained in \(p\), and \(p_1 \cdots p_n\) a product of finite number of primes which is contained in \(c\). Suppose now that \(\lambda\) is minimal. Then \(\lambda \neq 1\). For, let \(\lambda = 1\), then \(p_1 \leq c \leq p, p_i = e = p\). Hence \(c^{-1} = p^{-1} = e\), hence \(c = e\), and hence \(p = e\), a contradiction. Since \(p_1 \cdots p_n \leq c\), there exists \(p\), such that \(p_i \leq p, p_i = p\). Putting \(P = p_1 \cdots p_{i-1}, Q = p_{i+1} \cdots p_n\), we have \(c^{-1}P \cdot pQ \leq c^{-1}c = e\), and \(c^{-1}P \leq (pQ)^{-1}\). On the other hand, since \(pQ(pQ)^{-1} \leq e\), we have \(Q(pQ)^{-1} \leq p^{-1} = e\), and \((pQ)^{-1} \leq Q^{-1}\). Hence \(c^{-1}P \leq Q^{-1}\). This implies \(c^{-1}PQ \leq Q^{-1}Q \leq e, PQ \leq c^{-1}\) = \(e\), i.e. \(p_1 \cdots p_{i-1}p_{i+1} \cdots p_n \leq c\), we have a contradiction to the minimality of \(\lambda\).

**Theorem 1.** Let \(L\) be a residuated lattice with a maximally integral identity \(e\). Suppose that

1. The r.d.c.c. holds for integral elements of \(L\).
2. Any prime element is divisor-free.
3. Any prime element contains an element \(c\) such that \(c^{-1} = c\).

Then \(L\) forms a commutative cl-group, which is a direct product of infinite cyclic groups generated by prime elements.

**Proof.** \(aa^{-1} \leq e\) is evident. Since \((aa^{-1})(aa^{-1})^{-1} \leq e\), we have \(a^{-1} (aa^{-1})^{-1} \leq a^{-1}, (aa^{-1})^{-1} \leq e\). Hence \(aa^{-1} = e\). \(L\) forms therefore a cl-group. The other part of the theorem is easily obtained. Q.E.D.

It is easy to prove the converses of Theorems 1 and 2.6 [1]. Hence we obtain the following:

**Theorem 2.** Let \(L\) be a residuated lattice with a maximally integral identity. Suppose that any prime is divisor-free and contains an element \(c\) satisfying \(c^{-1} = c\). Then the following two conditions are equivalent.

1. The a.c.c. holds for integral elements of \(L\).
2. The r.d.c.c. holds for integral elements of \(L\).

By Theorem 4.5 in [1], we obtain

**Theorem 3.** Let \(\circ\) be a regular order in a semigroup. Suppose that \(\circ\) is maximal and any closed prime \(\circ\)-ideal is a maximal closed two-sided \(\circ\)-ideal. Then the followings are equivalent:

(A) The a.c.c. holds for closed integral \(\circ\)-ideals.
(B) The r.d.c.c. holds for closed integral \(\circ\)-ideals.

**References**
