17. On Hardy and Littlewood’s Theorem

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(Comm. by Z. SUETUNA, M.J.A., Feb. 12, 1957)

1. Let \( f(x) \) be an \( L \)-integrable function with period \( 2\pi \), and its Fourier series be

\[
1 \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right).
\]

A. Zygmund [1] has shown the following

**Theorem Z.** If \( f(x) \) belongs to Lip \( \alpha \) where \( 0 < \alpha \leq 1 \), then the series (1) is uniformly summable \( (C, -\alpha + \delta) \) to \( f(x) \) for every \( \delta > 0 \).

Later, Hardy and Littlewood [2] showed the following

**Theorem H.L.** If \( f(x) \) belongs to Lip \( (\alpha, p) \) where \( 0 < \alpha \leq 1 \) and \( \alpha p > 1 \), i.e.

\[
\left( \int_0^{2\pi} |f(x+h)-f(x)|^p \, dx \right)^{1/p} = O(h^\alpha)
\]
as \( h \to 0 \), then the series (1) is uniformly summable \( (C, -\alpha + \delta) \) to \( f(x) \) for every \( \delta > 0 \).

In this paper we shall improve the above theorem as follows:

**Theorem.** If \( f(x) \) is continuous in \( (0, 2\pi) \), and belongs to Lip \( (\alpha, 1/\alpha) \) where \( 0 < \alpha \leq 1 \), i.e.

\[
\int_0^{2\pi} |f(x+h)-f(x)|^{1/\alpha} \, dx = O(h)
\]
as \( h \to 0 \), then the series (1) is uniformly summable \( (C, -\alpha + \delta) \) to \( f(x) \) for every \( \delta > 0 \).

2. The proof\(^*)\) of our theorem is as follows. Let

\[
\varphi(t) = \varphi_\alpha(t) = f(x+t) + f(x-t) - 2f(x),
\]

then we have

\[
\varphi(t) \to 0 \text{ as } t \to 0 \text{ uniformly in } 0 \leq x \leq 2\pi,
\]
since \( f \) is continuous.

We denote the \( n \)-th \((C, \gamma)\) mean of the series (1) by \( \sigma_\alpha^n(x) \), then

\[
\sigma_\alpha^n(x) - f(x) = \frac{1}{\pi} \int_0^{2\pi} \varphi(t)K_\alpha^{-n}(t) \, dt
\]

\[
= \frac{1}{\pi} \int_0^{\pi/n} \frac{1}{\sin t} \int_{K/n}^\pi = I_1 + I_2
\]
say, where \( K_\alpha^n(t) \) is the \( n \)-th \((C, \gamma)\) Féjer kernel and

\[
|K_\alpha^{-n}(t)| \leq \frac{n}{1-\alpha} + \frac{1}{2} \text{ for } 0 \leq t \leq \pi,
\]

\(^*)\) The method of this proof has been suggested to me by Prof. G. Sunouchi.
and

\[ K_n^{-\alpha}(t) = \Re(e^{int}/A_n^{-\alpha}(1-e^{-it})^{1-\alpha}) + O(1/nt^2) \]
for \( 0 < t \leq \pi \).

By (2) and (3) it holds

\[ |I_1| < \varepsilon_n K \]
uniformly concerning \( x \), where \( \varepsilon_n > 0 \) and \( \varepsilon_n \to 0 \). And we see easily that, by (4), (2) and boundedness of \( f \),

\[ I_2 = \Re\left( \frac{1}{2\pi A_n^{-\alpha}} \int_{\pi/n}^{\pi} \frac{\varphi(t) - \varphi(t + \pi/n)}{(1-e^{-it})^{1-\alpha}} e^{int} \, dt \right) + O(1/K^{1-\alpha}), \]

where \( O \) is uniform concerning \( x \).

Replacing \( -\alpha \) by \( -\alpha + \delta \) we have

\[ |\sigma_n^{-\alpha+\delta}(x) - f(x)| < C_1 n^{1-\delta} \int_{\pi/n}^{\pi} |\varphi(t) - \varphi(t + \pi/n)| \, dt + C_2/K^{1-\alpha} + \varepsilon_n K, \]

where, and in succession, \( C \)'s are absolutely positive constants, not depending on \( x \).

First suppose that \( \alpha < 1 \), then since \( f \in \text{Lip}(\alpha, 1/\alpha) \) we have

\[ n^{1-\delta} \int_{\pi/n}^{\pi} |\varphi(t) - \varphi(t + \pi/n)| \, dt \]
\[ \leq n^{1-\delta} \left( \int_{0}^{\pi} |\varphi(t) - \varphi(t + \pi/n)|^{1/\delta} \, dt \right) \left( \int_{\pi/n}^{\pi} (1/t^{1-\alpha+\delta})^{1/(1-\delta)} \, dt \right)^{1-\alpha} \]
\[ \leq C_3 n^{1-\delta}(1/n^\delta)(n/K)^{\delta} = C_3/K^{\delta}. \]

In the case \( \alpha = 1 \), since \( f \in \text{Lip}(1, 1) \),

\[ n^{1-\delta} \int_{\pi/n}^{\pi} |\varphi(t) - \varphi(t + \pi/n)| \, dt \]
\[ \leq n^{1-\delta}(n/K)^{\delta} \int_{0}^{\pi} |\varphi(t) - \varphi(t + \pi/n)| \, dt \]
\[ \leq C_4 n^{1-\delta}(n/K)^{\delta}(1/n) = C_4/K^{\delta}. \]

Thus we have from (5)

\[ |\sigma_n^{-\alpha+\delta}(x) - f(x)| < C_5 K^{\delta} + C_2/K^{1-\alpha} + \varepsilon_n K, \]

With \( n \to \infty \) and then \( K \to \infty \) we get the desired result.

References
