111. A Note on Some Topological Spaces

By Shouro Kasahara
Kobe University

This short note has two purposes: one of them is to determine a topological space treated in [2], that is, it will be shown below that a Hausdorff space satisfying one of the conditions listed in Theorem 2 of [2] is nothing more than a finite set; and the other is to point out a more fundamental property of weakly compact spaces.\(^{*)}\n
Let \( \mathcal{O} = \{O_a\}_{a \in A} \) be a family of subsets of a topological space \( E \). We say that \( \mathcal{O} \) is point finite if each point of \( E \) belongs at most to a finite number of the members of \( \mathcal{O} \). The family \( \mathcal{O} \) is said to be locally finite if each point of \( E \) possesses a neighbourhood which intersects at most finitely many members of \( \mathcal{O} \); and \( \mathcal{O} \) is star finite if each member of \( \mathcal{O} \) intersects at most finitely many members of \( \mathcal{O} \). Moreover, \( \mathcal{O} \) is termed weakly locally finite if \( \mathcal{O} \) is locally finite as a family of subsets of the subspace \( \bigcup_{a \in A} O_a \) of \( E \) (i.e. if each point of \( \bigcup_{a \in A} O_a \) possesses a neighbourhood which intersects finitely many members of \( \mathcal{O} \)). If the set \( A \) of indices is a finite set, the family \( \mathcal{O} \) is called finite. Obviously, a star finite family is weakly locally finite, a locally finite family is weakly locally finite, and a weakly locally finite family is point finite.

**Theorem 1.** The following conditions on a Hausdorff space \( E \) are equivalent:

1. Every point finite open covering of \( E \) is finite.
2. Every point finite family of open sets of \( E \) is finite.
3. Every weakly locally finite family of open sets of \( E \) is finite.
4. Every star finite family of open sets of \( E \) is finite.
5. Every family of pairwise disjoint open sets of \( E \) is finite.
6. \( E \) is a finite set.

**Proof.** It will suffice to prove that (5) implies (6). To prove this, it is sufficient to show that each point of \( E \) is open. Suppose that there exists a point \( x \in E \) which is not open. Then, if \( V_x \) is a neighbourhood of \( x \), we can find a point \( x_1 \in V_x \) distinct from \( x \), and then we can choose disjoint open sets \( V_i \) and \( O_i \) such that \( x \in V_i \), \( x \in O_i \), and \( V_i \subseteq V_x, O_i \subseteq V_x \). Thus, by induction, it is easy to construct a sequence \( \{V_n\} \) of neighbourhoods of \( x \) and a sequence \( \{O_n\} \) of open

\(^{*)}\) For the definition of weakly compact space (espace faiblement compact), see [3 or 4].
sets such that $V_{n-1} \supseteq V_n$, $V_{n-1} \supseteq O_n$ and $V_n \cap O_n = \emptyset$ for every positive integer $n$. But, since the family $\{O_n\}$ consists of pairwise disjoint open sets, this contradicts the condition (5).

In a letter of June 22, 1957, to the author, Prof. S. Mardešić has kindly pointed out an error in [1] that though a characterization of pseudo-compact spaces is cited in [1] as a result of S. Mrówka, the truth is that it is due to J. Kernstan and not to S. Mrówka, and further he has kindly informed that the result obtained by S. Mrówka in this field is as follows: A space is pseudo-compact if and only if every locally finite open covering of it is finite.

The circumstance between these results (see also Theorem 1 of [3]) will become apparent in view of the following

**Theorem 2.** A topological space $E$ is weakly compact if and only if every locally finite family of open sets of $E$ is finite.

**Proof.** Since every family of pairwise disjoint open sets is whether locally finite or not, the "only if" part of the theorem is evident. In order to prove the "if" part of the theorem, let us suppose that there exists a locally finite family $\{O_a\}_{a \in A}$ consisting of infinitely many open sets. Then, for any point $x_1$ in $\bigcup_{a \in A} O_a$, we can find an open neighbourhood $U_i$ which intersects only a finite number of $O_a$'s. Suppose now that we have obtained pairwise disjoint open sets, $U_1, U_2, \ldots, U_n$ such that each $U_i$ intersects only a finite number of $O_a$'s. Then since the power of $A$ is infinite, there exists a member $O_a^{n+1}$ of the family for which we have $U_{a+1} \cap U_i = \emptyset$ for $i = 1, 2, \ldots, n$.

Let $x_{n+1}$ be a point of $O_a^{n+1}$, and let $U_{n+1} \subseteq O_a^{n+1}$ be an open neighbourhood of $x_{n+1}$ which intersects finitely many members of $\{O_a\}_{a \in A}$. It is clear that the open set $U_{n+1}$ is disjoint from each $U_i, i = 1, 2, \ldots, n$. We have thus a sequence $\{U_n\}$ of pairwise disjoint open sets. But, from the assumption that the space $E$ is weakly compact, it follows that $\{U_n\}$ is not locally finite. Since $U_i \subseteq O_a^n$ for any positive integer $n$, this is absurd. Therefore the family $\{O_a\}_{a \in A}$ must be finite.

**References**


