81. Relations between Solutions of Parabolic and Elliptic Differential Equations

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(Comm. by K. KUNUGI, M.J.A., June 12, 1958)

In this note we shall show that under some conditions the solution 
\[ u(x, t) \]
converges to a solution 
\[ v(x) \]
as \( t \to \infty \).

Let \( G \) be a domain which is regular for Laplace's equation\(^1\) in
the \( m \)-dimensional Euclidean space, and let \( I' \) be the boundary of \( G \).
Set \( D=G \times (0, \infty) \) and \( B=I' \times [0, \infty) \). We remark that \( D \) is regular
for the heat equation\(^2\) and therefore regular for the equation \((E)\) below.\(^3\)

Now, let \( \Box \) and \( \triangle \) be the generalized heat operator\(^4\) and the
generalized Laplacian operator respectively, i.e.
\[
\Box u(x, t) = \lim_{r \to 0} \frac{(n+2)}{m \pi^{\frac{n}{2}} r^2} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \cdots \int_0^{\frac{\pi}{2}} \{u(\xi, \tau) - u(x, t)}] \sin^{n-1} \theta \times \cos \theta (\log \cosec \theta)^{\frac{m}{2}} \int dq_1 \cdots dq_{m-1} d\theta
\]
and
\[
\triangle u(x) = \lim_{r \to 0} \frac{2 \cdot I'(m+1)}{m \pi^m r^2} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \cdots \int_0^{\frac{\pi}{2}} \{u(\xi) - u(x)}] \int dq_1 \cdots dq_{m-1},
\]
where in the first expression, \((\xi, \tau) = (\xi_1, \cdots, \xi_m, \tau)\) with
\[
\xi_i = x_i + 2r \sqrt{m} \sin \nu / \log \cosec \theta \eta_i \quad (i = 1, \cdots, m)
\]

1) This means that the 1st boundary value problem of Laplace's equation for \( G \)
is always solvable for any continuous data on \( I' \).
2) "Regular for the heat equation" means that the 1st boundary value problem
of the heat equation for \( D \) is always solvable for any continuous data on \( G \sim B \). \( D \)
is regular for the heat equation if and only if \( G \) is regular for Laplace's equation.
For the proof, see "On the regularity of domains for parabolic equations", Proc.
3) It was proved in [1, p. 626] that a \( p \)-domain is regular for \((E)\) if and only if
it is regular for the heat equation.
4) See [1, p. 627], in which we used the symbol \( \Box \) instead of \( \Box \).
\[ \tau = t - r^2 \sin^2 \theta \quad \left( 0 \leq \theta \leq \frac{\pi}{2} \right) \]
and in the second expression, \( (\xi) = [\xi_1, \ldots, \xi_m] \) with
\[ \xi_i = x_i + r \eta_i \quad (i = 1, \ldots, m). \]

In both cases,
\[
\begin{align*}
\eta_1 &= \cos \varphi_1 \cos \varphi_2 \cdots \cos \varphi_{m-2} \cos \varphi_{m-1} \\
\eta_2 &= \cos \varphi_1 \cos \varphi_2 \cdots \cos \varphi_{m-2} \sin \varphi_{m-1} \\
\eta_3 &= \cos \varphi_1 \cos \varphi_2 \cdots \sin \varphi_{m-2} \\
\cdots & \cdots \cdots \cdots \\
\eta_{m-1} &= \cos \varphi_1 \sin \varphi_2 \\
\eta_m &= \sin \varphi_1 \\
&\quad \left( -\frac{\pi}{2} \leq \varphi_i \leq \frac{\pi}{2}, \ i = 1, \ldots, m-2; \ 0 \leq \varphi_{m-1} \leq 2\pi \right)
\end{align*}
\]

and
\[
J = \det \begin{vmatrix}
\eta_1 & \eta_2 & \cdots & \eta_m \\
\frac{\partial \eta_1}{\partial \varphi_1} & \frac{\partial \eta_2}{\partial \varphi_1} & \cdots & \frac{\partial \eta_m}{\partial \varphi_1} \\
\frac{\partial \eta_1}{\partial \varphi_{m-1}} & \frac{\partial \eta_2}{\partial \varphi_{m-1}} & \cdots & \frac{\partial \eta_m}{\partial \varphi_{m-1}} \\
\end{vmatrix}
\]

These operators have the following properties:

(i) If \( u(x, t) \) and \( u(x) \) are functions in the class \( C^2 \),
\[
\Box u(x, t) = \sum_{i=1}^{m} \frac{\partial^2 u(x, t)}{\partial x_i^2} - \frac{\partial u(x, t)}{\partial t}
\]
and
\[
\triangle u(x) = \sum_{i=1}^{m} \frac{\partial^2 u(x)}{\partial x_i^2}.
\]

(ii) If we operate \( \Box \) to a function \( u(x) \) which does not depend on \( t \), we have
\[
\Box u(x) = \triangle u(x).
\]

Consider the following two equations:

(E1) \( \Box u = f(x, t, u) \quad x \in G, \ t \geq 0, \)

(E2) \( \triangle v = \tilde{f}(x, v) \quad x \in G, \)

where \( f(x, t, u) \) and \( \tilde{f}(x, v) \) are continuous functions on \( D \times (-\infty, \infty) \) and \( G \times (-\infty, \infty) \) respectively, quasi-bounded with respect to \( u \) and \( v \) and non-decreasing with respect to \( u \) and \( v \).

Let \( g(x) \) be a continuous function on \( G \cup \Gamma \) and \( \varphi(x, t) \) be a continuous function on \( B \) and moreover \( \varphi(x, 0) = g(x) \) for \( x \in \Gamma \). Let \( u(x, t) \) be a solution of \( (E_1) \) which is continuous on \( D \cup \Gamma \cup B \) and which satisfies the boundary condition \( u(x, 0) = g(x) \ (x \in G) \) and \( u(x, t) = \varphi(x, t) \ (x \in \Gamma, \ t \geq 0) \). Assume that \( \varphi(x, t) \) converges uniformly on \( \Gamma \) to a

5) We say that a function \( f(p, q) \) defined on \( E \times F \) is quasi-bounded with respect to \( q \) if \( f(p, q) \) is bounded on \( E \times K \), where \( K \) is any compact set in \( F \).

6) These solutions exist. See [1] and [2].
function \( \varphi(\overline{x}) \) as \( t \to \infty \). (Then \( \varphi(\overline{x}) \) is again a continuous function on \( I' \).) Let \( v(x) \) be a solution\(^7\) of \( (E_2) \) which is continuous on \( G \backslash I' \) and satisfies \( v(\overline{x}) = \varphi(\overline{x}) \) on \( I' \).

Finally assume that, for any \( U > 0 \), \( f(x, t, u) \) converges uniformly to \( \bar{f}(x, u) \) on the set \( \{(x, u); x \in G, |u| \leq U\} \) as \( t \to \infty \).

Under these assumptions, \( u(x, t) \) converges uniformly to \( v(x) \) on \( G \backslash I' \) as \( t \to \infty \).

Proof. For any \( \varepsilon > 0 \), there exists \( T_1 > 0 \) such that \( |\varphi(\overline{x}, t) - \varphi(\overline{x})| < \varepsilon \) for \( t \geq T_1 \). Set \( M_0 = \max \{ |g(x)|; x \in G \backslash I' \} \), \( M_1 = \max \{ |\varphi(\overline{x}, t)|; x \in I', 0 \leq t \leq T_1 \} \) and \( M_2 = \max \{ |v(x)|; x \in G \backslash I' \} \). By the assumption above we can find a constant \( T_2 > 0 \) such that

\[
|f(x, t, v(x)) - \bar{f}(x, v(x))| < \varepsilon
\]

for \( x \in G, t \geq T_2 \). Set \( M_3 = \sup \{ |f(x, t, v(x)) - \bar{f}(x, v(x))|; x \in G, 0 \leq t \leq T_2 \} \). Let \( \psi(x) \) be a solution of \( \Delta \psi = -1 \) such that \( \psi(x) \) is continuous on \( G \backslash I' \) and vanishes on \( I' \). Then there exists a constant \( \mathfrak{P} \) such that \( 0 \leq \psi(x) \leq \mathfrak{P} \), hence we can take a constant \( \alpha > 0 \) such that \(-1 + \alpha(1 + \mathfrak{P}) < -\frac{1}{2}\). Finally, let \( M > 0 \) be a constant such that \( (i) \frac{1}{2} Me^{-\alpha t} > M_3 \),

(ii) \( Me^{-\alpha t} > M_1 + M_2 \) and (iii) \( M > M_0 + M_2 \).

Consider the function \( Me^{-\alpha t} + \varepsilon \). Then, we have

\[
|\varphi(\overline{x}, t) - \varphi(\overline{x})| < Me^{-\alpha t} + \varepsilon \quad x \in I', t \geq 0.
\]

Now, let \( v_1(x, t) \) be a solution of the equation:

\[
\sum_{i=1}^{m} \frac{\partial^2 v_1(x, t)}{\partial x_i^2} = -Me^{-\alpha t} - \varepsilon,
\]

and suppose that \( v_1(x, t) \) is continuous on \( G \backslash I' \) and admits the boundary value \( Me^{-\alpha t} + \varepsilon \) on \( I' \). Then we have

\[
v_1(x, t) = Me^{-\alpha t} + \varepsilon + \psi(x)(Me^{-\alpha t} + \varepsilon)
= Me^{-\alpha t}(1 + \psi(x)) + \varepsilon(1 + \psi(x))
= (Me^{-\alpha t} + \varepsilon)(1 + \psi(x)).
\]

Set \( V(x, t) = v(x) + v_1(x, t) \), then

\[
\bigcirc V(x, t) = \Delta v(x) + \sum_{i=1}^{m} \frac{\partial^2 v_1(x, t)}{\partial x_i^2} - \frac{\partial v_1(x, t)}{\partial t}
= \bar{f}(x, v(x)) + (-Me^{-\alpha t} - \varepsilon) + \alpha Me^{-\alpha t}(1 + \psi(x))
= \bar{f}(x, v(x)) + \varepsilon + \alpha Me^{-\alpha t}(1 + \psi(x)).
\]

Now, for \( u > V(x, t) \) we have

\[
f(x, t, u) - \bigcirc V(x, t) \geq f(x, t, v(x)) - \bar{f}(x, v(x)) + \varepsilon - Me^{-\alpha t}(-1 + \alpha(1 + \psi(x))).
\]

Since \( f(x, t, v(x)) - \bar{f}(x, v(x)) > -M_3 \) and \( -Me^{-\alpha t}(-1 + \alpha(1 + \psi(x)) > M_3 \)

for \( 0 \leq t \leq T_2 \), we have

\[
f(x, t, u) - \bigcirc V(x, t) > 0
\]

for \( 0 \leq t \leq T_2 \). For \( t > T_2 \), since \( f(x, t, v(x)) - \bar{f}(x, v(x)) > -\varepsilon \), we have

\[8\text{) It is sufficient for our proof to assume that } f(x, t, v(x)) \text{ converges uniformly to } \bar{f}(x, v(x)) \text{ on } G \text{ as } t \to \infty.\]
\[ f(x, t, u) - \varphi V(x, t) > 0. \]

Consequently, if \( u > V(x, t) \), \( x \in G \) and \( t > 0 \), then we obtain
\[ f(x, t, u) - \varphi V(x, t) > 0. \]

Next, on the boundary \( B \), since \( \varphi(x, t) \leq \varphi(x) + M e^{-at} + \varepsilon \), we have
\[ u(x, t) = \varphi(x, t) \leq \varphi(x) + (M e^{-at} + \varepsilon)(1 + \psi(x)) = V(x, t). \]

On \( G \), the rest part of the boundary of \( D \),
\[ u(x, 0) = g(x) < M_0 < M - M_2 v(x) + (1 + \psi(x))(M + \varepsilon) \]
implies \( V(x, 0) \geq u(x, 0) \). Hence, on the whole boundary of \( D \), we have
\[ V(x, t) \geq u(x, t). \]

Therefore by the comparison theorem,\(^9\) we have
\[ u(x, t) \leq V(x, t) = v(x) + v_1(x, t) \]
on \( D \sim G \sim B \). Similarly we have \( v(x) - v_1(x, t) \leq u(x, t) \), and consequently
\[ |u(x, t) - v(x)| \leq v_1(x, t) \]
on \( D \sim G \sim B \).

Since \( v_1(x, t) = (M e^{-at} + \varepsilon)(1 + \psi(x)) \), there exists a constant \( T_3 > 0 \) such that \( |v_1(x, t)| \leq 2(1 + \psi) \varepsilon \) for \( x \in G \sim \Gamma \) and \( t \geq T_3 \). Thus \( u(x, t) \) converges uniformly to \( v(x) \) on \( G \sim \Gamma \). This completes the proof.

**Corollary 1.** Assume that moreover \( f(x, t, 0) = 0 \). Then, the solution of (E1) which admits \( g(x) \) on \( G \) (where \( g(x) = 0 \) for \( x \in \Gamma \)) and which vanishes on \( B \) converges uniformly to zero on \( G \sim \Gamma \).

This shows that the solution is asymptotically stable.

**Corollary 2.** If \( \varphi(x, t) \) converges uniformly to \( \varphi(x) \) on \( \Gamma \), the solution of the heat equation which admits \( \varphi(x, t) \) on \( B \) and which admits \( g(x) \) on \( G \) converges uniformly to the solution of Laplace's equation which admits \( \varphi(x) \) on \( \Gamma \).

This means that the solution of the heat equation converges to the steady state solution.\(^{10}\)

**References**


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9) Theorem 2.1 [1, p. 533].

10) See also W. Fulks: A note on the steady state solution of the heat equation, Proc. Amer. Math. Soc., 7 (1956). He assumed that \( \varphi(x, t) \) is monotone increasing with \( t \). This assumption plays essential role in his proof but our proof does not need it.