139. On the Cohomology Groups of \( \mathbb{p} \)-adic Number Fields

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In the present note we shall study the cohomology groups of the ring of all \( \mathbb{p} \)-integers of a \( \mathbb{p} \)-adic field.

Let \( K \) be a \( \mathbb{p} \)-adic number field and let \( L \) be a finite separable extension field over \( K \). More generally, let \( K \) be a complete field by a discrete valuation and let \( L \) be a finite separable extension field over \( K \) with separable residue class field. Let \( R \) and \( A \) be the rings of all \( \mathbb{p} \)-integers of \( K \) and \( L \), respectively. Then \( A \) has a minimal basis over \( R \), i.e.

\[
A = R + R\theta + \cdots + R\theta^{n-1},
\]

where \( 1, \theta, \ldots, \theta^{n-1} \) are linearly independent over \( R \) [1]. Let \( f(x) = 0 \) be the equation of \( \theta \) in \( R \).

We shall consider \( A \) as an algebra over \( R \) and construct a \( \Lambda^* \)-projective resolution over \( A \) which is suitable for our purpose.

Let

\[
f(x) = (x-\theta)g(x), \quad g(x) = x^{n-1} + \left( \sum_{j=0}^{n-2} \theta^j x^{n-2-j} \right)
\]

be the decomposition of \( f(x) \) in \( A \). We put

\[
g_0(\theta) = \sum_{i,j} b_{ij} \theta^i \otimes \theta^j \quad \text{and} \quad \Delta \theta = \theta \otimes 1 - 1 \otimes \theta
\]

in \( \Lambda^* = A \otimes_R A \).

Lemma

Let \( \sum \lambda \otimes \mu \) be any element in \( \Lambda^* \). Then

\[
(\sum \lambda \otimes \mu)(\theta \otimes 1 - 1 \otimes \theta) = 0 \quad \text{if and only if} \quad \sum \lambda \otimes \mu \in \Lambda^* \cdot g_0(\theta);
\]

\[
(\sum \lambda \otimes \mu) \cdot g_0(\theta) = 0 \quad \text{if and only if} \quad \sum \lambda \otimes \mu \in \Lambda^* (\theta \otimes 1 - 1 \otimes \theta).
\]

Proof. Since we have a ring isomorphism

\[
\Lambda \otimes_R A \cong A[x]/(f(x)),
\]

\[
\theta \otimes 1 - 1 \otimes \theta \leftrightarrow x - \theta \quad \text{mod} \quad (f(x)),
\]

\[
g_0(\theta) \leftrightarrow g(x) \quad \text{mod} \quad (f(x)),
\]

we shall calculate in the right hand side. We take polynomials of degree less than \( n \) as the uniquely determined representatives of the classes mod \( f(x) \). If \( (x-\theta)h(x) \equiv 0 \) mod \( f(x) \), deg \( h(x) \leq n-1 \), then dividing \( h(x) \) by \( g(x) \) we have \( h(x) = \alpha g(x) + s(x) \), deg \( s(x) \leq n-2 \); so \( s(x)(x-\theta) \equiv 0 \) mod \( f(x) \). Therefore \( s(x) = 0 \), \( h(x) = \alpha g(x) \). Similarly, if \( g(x)h(x) \equiv 0 \) mod \( f(x) \), then \( h(x) = (x-\theta)h_0(x) \).

1) Since \( A \) is commutative, \( A^* = A \) and we shall drop the sign \( \ast \).
The kernel of the augmentation \( \varepsilon : \mathcal{A}^e \to A \), \( \varepsilon(\lambda \otimes \mu) = \lambda \mu \) is \( \mathcal{A}'(\theta \otimes 1 - 1 \otimes \theta) \).

Proof. Since \( A \) is commutative, \( \varepsilon \) is a ring homomorphism. So that \( \mathcal{A}'(\theta \otimes 1 - 1 \otimes \theta) \) is contained in the kernel of \( \varepsilon \). Conversely, if 
\[
\varepsilon(\sum_{i,j} c_{ij} \theta^i \otimes \theta^j) = 0,
\]
then from 
\[
\sum_{i,j} c_{ij} \theta^i \otimes \theta^j = 0
\]
we have \( \varepsilon((\sum c_{ij} \theta^{i+j}) \otimes 1) = 0 \), which proves the assertion.

Now we consider the following \( \mathcal{A}^e \)-resolution over \( A \):
\[
\cdots \xrightarrow{d_4} \mathcal{A}^e \xrightarrow{d_3} \mathcal{A}^e \xrightarrow{d_2} \mathcal{A}^e \xrightarrow{d_1} \mathcal{A}^e \xrightarrow{\varepsilon} A \longrightarrow 0
\]
where \( \varepsilon : \mathcal{A}^e \to A \), \( \varepsilon(\sum \lambda \otimes \mu) = \sum \lambda \mu \).

This is \( \mathcal{A}^e \)-free and, by the above lemma, acyclic.

To calculate \( H^n(A, A) \) and \( H_n(A, A) \) for any \( \mathcal{A}^e \) module \( A \), we consider the complex
\[
\cdots \xleftarrow{\delta_i} \text{Hom}_{\mathcal{A}^e}(A^e, A) \xrightarrow{\delta_i} \xrightarrow{\delta_i} \text{Hom}_{\mathcal{A}^e}(A^e, A)
\]
where \( \delta_i \) and \( \delta_i \) are induced homomorphisms of \( \delta_i \). Considering the isomorphisms
\[
\text{Hom}_{\mathcal{A}^e}(A^e, A) \cong A, \quad A \otimes_{\mathcal{A}^e} A^e \cong A,
\]
we may translate \( \delta_i \) and \( \delta_i \) into the endomorphisms of \( A \)
\[
\delta_{2r+1}(a) = a \theta - \theta a, \quad \delta_{2r+1}(a) = \theta a - a \theta,
\]
for \( a \in A \). Thus we have

Theorem
\[
H^{2r+1}(A, A) \cong A g_{q_{2r+1}}(A^e, A) \quad H^{2r+2}(A, A) \cong A_{q_{2r+1}}(A^e, A),
\]
for \( r \geq 0 \), where
\[
A^e = \{ a \in A | \square a = 0 \}, \quad A^e = \{ \square a | a \in A \}
\]
for any two sided \( A \) module \( A \) (considered as left \( \mathcal{A}^e \) module).

Corollary
\[
H^{n+2}(A, A) \cong H^n(A, A)
\]
for \( n \geq 1 \).

Theorem
If \( \theta a = a \theta \) for any \( a \) in \( A \), then
\[
H^{2r+1}(A, A) \cong H_{2r+2}(A, A) \cong A_{q_{2r+1}}(A^e, A),
\]
for \( r \geq 0 \).
Proof. In this case \( g_e(\theta) \cdot a = g(\theta)a \) and
\[
g(\theta) = (\theta - \theta') \cdots (\theta - \theta^{n-1}) = f'(\theta).
\]

The corollary of this note may be extended to the global case. Let \( K \) and \( L \) be the algebraic number fields, \( R \) and \( A \) the rings of all integers of \( K \) and \( L \) respectively. Then for any \( A \)-finitely generated module \( A \) we have
\[
H_{n+2}(A, A) \cong H_n(A, A)
\]
\[
H_{n+2}(A, A) = H_n(A, A)
\]
for \( n \geq 1 \). We may prove it by reducing it to the \( p \)-component and by using the above corollary.

References