22. An Abstract Analyticity in Time for Solutions of a Diffusion Equation

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1. Introduction and the result. Consider an equation of evolution

\[ \frac{\partial u}{\partial t} = Au, \quad t > 0, \]

where the differential operator

\[ A = a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + b^i(x) \frac{\partial}{\partial x_i} + c(x) \]

is elliptic in a connected domain \( G \) of an \( m \)-dimensional euclidean space \( E^m \). Under certain conditions upon the coefficients \( a, b \) and \( c \) of \( A \), we can specify a linear subspace \( D \) of \( L_2(G) \) with the following three properties.

(i) The functions \( \in D \) are \( C^\infty \) in \( G \), and \( D \) is \( L_2(G) \)-dense in \( L_2(G) \) such that \( Af \in L_2(G) \) for \( f \in D \).

(ii) If we consider \( A \) as an operator on \( D \subseteq L_2(G) \) into \( L_2(G) \), then \( A \) admits, in \( L_2(G) \), the smallest closed extension \( \hat{A} \).

(iii) \( \hat{A} \) is the infinitesimal generator of a semi-group \( T_t \) of normal type in \( L_2(G) \) such that, for any \( f \in L_2(G) \), \( u(t, x) = (T_t f)(x) \) is a solution of (1.1) with the initial condition

\[ L_2(G)\text{-lim}_{t \to 0^+} u(t, x) = f(x) \]

satisfying the “forward and backward unique continuation property”:

(1.3) If, for a fixed \( t_0 > 0 \), \( u(t_0, x) = 0 \) on an open set \( G_0 \subseteq G \), then \( u(t, x) = 0 \) for every \( t > 0 \) and every \( x \in G_0 \).

The proof of (1.3) is based upon the fact that \( T_t f \) is an \( L_2(G) \)-valued abstract analytic function of \( t \) in a certain sector of the complex plane which contains the positive \( t \)-axis in its interior and with \( t=0 \) as its vertex. Such abstract analyticity in time is implied by the estimate (2.11) below of the resolvent of \( \hat{A} \).

Our result (1.3) gives a partial answer to a conjecture proposed by S. Ito and H. Yamabe [2]. Actually, our solution \( u(t, x) = (T_t f)(x) \) enjoys the “unique continuation property”:

(1.3') If, for a fixed \( t_0 > 0 \), \( u(t_0, x) = 0 \) on an open set \( G_0 \subseteq G \), then \( u(t, x) = 0 \) for every \( t > 0 \) and every \( x \in G \).

1) This estimate was given in the author’s lecture at Yale University in the fall of 1958.
This may be proved by combining (1.3) with the “space-like unique
continuation theorem for solutions of parabolic equations” obtained
recently by S. Mizohata [3]. Thus we obtain another proof of the
unique continuation theorem of S. Ito and H. Yamabe [2].

2. The proof of the result. For the sake of simplicity of ex-
position, we shall be concerned with the case G=E^m. We assume
that the real-valued coefficients a, b and c are C^\infty in E^m and that
\[ a^{ij}(x) \] and its first and second partials, \( b^i(x) \) and its first partials
and c(x) are, in absolute values, all bounded on E^m by a positive
constant \( \beta \).

Thus the strict ellipticity of \( A \) implies the existence of two positive
constants \( \gamma \) and \( \delta \) such that
\[
\gamma \sum_{j=1}^{m} \xi_j^2 \geq a^{ij}(x)\xi_i\xi_j \geq \delta \sum_{j=1}^{m} \xi_j^2 \quad \text{on } E^m
\]
for any real vector \((\xi_1, \xi_2, \ldots, \xi_m)\).

Let \( H_1=H_1(E^m) \) be the space of complex-valued C^\infty functions
\( f(x)=f(x_1, \ldots, x_m) \) in E^m for which
\[
\left\| f \right\| = \left( \int_{E^m} |f(x)|^2 dx + \sum_{j=1}^{m} \int_{E^m} |f_{x_j}(x)|^2 dx \right)^{1/2} < \infty,
\]
and let \( \hat{H}_1=L_2(E^m)=L_2 \) be the completion of \( H_1 \) with respect to the
norm
\[
\left\| f \right\| = \left( \int_{E^m} |f(x)|^2 dx \right)^{1/2}.
\]
We denote by RH_1 (and RL_2) the totality of real-valued functions
belonging to H_1 (and to L_2).

Lemma. There exist two positive constants \( \alpha_0 \) and \( \beta_0 \) such that,
for any \( f \in RH_1 \), the equation
\[
\alpha u - Au = f, \quad \alpha > \max (\alpha_0, \delta + \beta_0),
\]
admits a uniquely determined solution \( u(x)=u_f(x) \in RH_1 \), and we have
the estimate
\[
\left\| u_f \right\| \leq (\alpha - \delta - \beta_0)^{-1} \left\| f \right\|.
\]

Proof. The existence of the solution \( u_f \in RH_1 \) for sufficiently large
\( \alpha \) is proved in K. Yosida [4]. If we denote by \( (f, g) \) the inner prod-
uct \( \int_{E^m} f(x)g(x)dx \), then for any \( u \in RH_1 \),
\[
\left\| (\alpha I-A)u \right\| \cdot \left\| u \right\| \geq \left| (\alpha I-A)u, u \right|
\]
by Schwarz inequality. By partial integration, we have (see K. Yosida
[4])

2) For, these two authors treat the case where \( \hat{A} \) is self-adjoint with its spectrum
lying on negative real axis, and the estimate (2.11) is clear for such operator \( \hat{A} \).

3) If G is a bounded domain of E^m, the method of the following proof may be
modified so as to apply to the case where \( A \) is an elliptic differential operator of 2n-order
(\( n>1 \)).
\[(\alpha I - A)u, u\rangle = \alpha \|u\|^2 + \int_{\Omega} a^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \, dx + \int_{\Omega} \frac{\partial a^{ij}}{\partial x_i} \frac{\partial u}{\partial x_j} u \, dx - \int_{\Omega} b^i \frac{\partial u}{\partial x_i} u \, dx - \int_{\Omega} c u u \, dx.
\]

(2.8)

Hence we have, by (2.1) - (2.2) and the inequality \(|\varepsilon^2| \leq 2^{-1}(|\varepsilon|^2 + |\eta|^2)\),

\[
((\alpha I - A)u, u) \geq \alpha \|u\|^2 + \frac{d}{2} \|u\|^2 - \frac{d}{2} \|u\|^2
\]

(2.9)

\[
- m_0 \nu \|u\|^2 + \nu^{-1} m \|u\|^2 + m^{-1} \|u\|^2
\]

for any \(\nu > 0\). Thus we have (2.6) from (2.7), by taking \(\nu > 0\) so small that \((d - m_0 \nu) > 0\) and \(\beta_0 = m_0 \beta(m_0^{-1} - \nu + m^{-1}) > 0\).

**Corollary.** Let us consider \(A\) as an operator defined on \(\{f; f \in RH, Af \in RH\} \subseteq RL\) into \(RL\). Then the smallest closed extension \(\tilde{A}\), in \(RL\), of \(A\) satisfies the condition that, for \(\alpha > \max (\alpha_0, \delta + \beta_0)\), the inverse \((\alpha I - \tilde{A})^{-1}\) exists as a bounded linear operator defined on \(RL\) into \(RL\) with the estimate

\[
\|((\alpha I - \tilde{A})^{-1}) \| \leq (\alpha - \delta - \beta_0)^{-1}.
\]

**Theorem 1.** If we consider \(A\) as an operator on \(\{f; f \in H, Af \in H\} \subseteq L\) into \(L\), then the smallest closed extension \(\tilde{A}\), in \(L\), of \(A\) is the infinitesimal generator of a semi-group \(T_t\) in \(L\) which is strongly continuous in \(t\), \(\|T_t\| \leq \exp ((\delta + \beta_0)t)\) and such that

\[
\lim_{|\tau| \to \infty} |\tau| \cdot \|((\alpha + \sqrt{-1}\tau)I - A)^{-1}\| < \infty.
\]

**Proof.** By the lemma and the reality of the coefficients of \(A\), we see that the range \((\alpha I - A)f, H\) is, for \(\alpha > \max (\alpha_0, \delta + \beta_0)\), \(L\)-dense in \(L\). Moreover we have, for \((u + \sqrt{-1}v) \in H\),

\[
\|(\alpha I - A)(u + \sqrt{-1}v)|^2 = \|(\alpha I - A)u|^2 + \|(\alpha I - A)v|^2
\]

\[
\geq (\alpha - \delta - \beta_0)^2 \|u\|^2 + (\alpha - \delta - \beta_0)^2 \|v\|^2.
\]

Thus \((\alpha I - \tilde{A})^{-1}\) is a bounded linear operator on \(L\) into \(L\) satisfying

\[
\|((\alpha I - \tilde{A})^{-1}) \| \leq (\alpha - \delta - \beta_0)^{-1}.
\]

Hence the first part of the theorem is proved (see E. Hille-R. S. Phillips [1] or K. Yosida [5]). We have to show that (2.11) holds good. We have, for \(w \in H, \alpha > \max (\alpha_0, \delta + \beta_0)\),

\[
\|(\alpha + \sqrt{-1}\tau)I - A)w\| \cdot \|w\| \geq \|((\alpha + \sqrt{-1}\tau)I - A)w, w\|
\]

As in (2.9), we obtain

\[
\text{Real Part} \left( \left((\alpha + \sqrt{-1}\tau)I - A\right)w, w\right) = \alpha \|w\|^2 + \text{Real Part} \left( \int a^{ij} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} \, dx + \int \frac{\partial a^{ij}}{\partial x_i} \frac{\partial w}{\partial x_j} \, dx - \int b^i \frac{\partial w}{\partial x_i} \, dx - \int c w \, dx \right)
\]

\[
\geq (\alpha - \delta - \beta_0) \|w\|^2 + (\delta - m_0 \beta_0) \|w\|^2.
\]
Similarly we have
\[ |\Im \langle (\alpha + \sqrt{-1} \tau)I - A \rangle w, w \rangle | \geq |\tau | \cdot ||w||^2 - m_\beta ||w||^2 = |(\tau | - m_\beta ||w||^2 - m_\beta ||w||^2|.
\]
If we assume that there exists \( w \in H, ||w|| \neq 0, \) such that
\[ |\Im \langle (\alpha + \sqrt{-1} \tau)I - A \rangle w, w \rangle | \leq 2^{-1} ||(\tau | - m_\beta ||w||^2|^2
\]
for sufficiently large \( \tau \) (or for sufficiently large \(-\tau\)), then, for such large \( \tau \) (or \(-\tau\)),
\[ m_\beta ||w||^2 \geq 2^{-1} ||(\tau | - m_\beta ||w||^2
\]
Hence, for such large \( \tau \) (or \(-\tau\)),
\[ |\Re \langle (\alpha + \sqrt{-1} \tau)I - A \rangle w, w \rangle | \geq (\delta - m_\beta) \frac{(||\tau | - m_\beta ||w||^2)}{2m_\beta}.\]
Thus (2.11) is proved.

**Theorem 2.** The semi-group \( T_t \) is, for \( t > 0, \) strongly differentiable in \( t \) any number of times. Actually, if we denote by \( T_t^{(k)} \) the \( k \)-th strong derivative of \( T_t \) with respect to \( t \), then there exists a positive constant \( \varepsilon \) such that, for any \( t > 0, \) the sequence of operators
\[ \sum_{k=0}^{n} (k!)^{-1} (\lambda - t)^{k} T_t^{(k)} \]
is, as \( n \uparrow \infty, \) convergent in the sense of the norm of operators when
(2.12) \[ |\lambda - t| < \varepsilon t.\]

**Proof.** See K. Yosida [6].

**Corollary.** For any \( f \in L_2, u(t, x) = (Tf)(x) \) is infinitely differentiable in \( t > 0 \) and \( x \in E^m \) and satisfies the Cauchy problem (1.1) - (1.1)'.

**Proof.** If we apply, in the sense of the distribution of L. Schwartz, the elliptic differential operator
\[ \frac{\partial^2}{\partial t^2} + A \]
any number of times to \( u(t, x), \) then the result is locally square integrable in the product space \((0 < t < \infty) \times E^m \). Thus \( u(t, x) \) is equivalent to a function which is \( C^\infty \) in \((0 < t < \infty) \times E^m \). See, for the details, K. Yosida [4].

**Proof of 1.3.** Since \( T_t^{(k)} = A^k T_t \), we have, by Theorem 2,
\[ \lim_{n \to \infty} ||T_{t_0 + h} f - \sum_{k=0}^{n} (k!)^{-1} h^k A^k T_{t_0} f|| = 0\]
for sufficiently small \( h. \) Hence there exists a sequence \( \{n'\} \) of natural numbers such that
\[ u(t_0 + h, x) = \lim \sum_{n' \to \infty} (k!)^{-1} h^k A^k u(t_0, x) \quad \text{for almost all } x \in E^m.\]

By the hypothesis in (1.3), we have \( A^k u(t_0, x) = 0 \) in \( G_0, \) and hence \( u(t_0 + h, x) = 0 \) in \( G_0. \) Repeating the process we see that \( u(t, x) = 0 \) for every \( t > 0 \) and every \( x \in G_0. \)

4) The "if" part of Theorem 2 in K. Yosida [6] must be corrected as: if
\[ \lim_{|\tau| \to \infty} |\tau | \cdot ||R(1 + i\tau, A) || = 0, \] then \( T_t \) exists for every \( t > 0. \)
References