102. On Compactness of Weak Topologies

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Let $R$ be a space and $a_i$ ($\lambda \in \Lambda$) a system of mappings of $R$ into topological spaces $S_i$ with neighbourhood systems $\mathcal{N}_i$ ($\lambda \in \Lambda$). Concerning the weak topology of $R$ by $a_i$ ($\lambda \in \Lambda$), i.e. the weakest topology of $R$ for which all $a_i$ ($\lambda \in \Lambda$) are continuous, we have (H. Nakano: Topology and Linear Topological Spaces, Tokyo (1951), §19, Theorem 4. This book will be denoted by TLTS):

Theorem 1. If all $S_i$ ($\lambda \in \Lambda$) are compact Hausdorff spaces, then, in order that the weak topology of $R$ be compact, it is necessary and sufficient that for any system of points $a_i \in S_i$ ($\lambda \in \Lambda$) subject to the condition

$$\bigcap_{\lambda \in \Lambda} a_i^{-1}(U_{\lambda}) \neq \emptyset$$

for every finite number of open sets $a_i \in U_{\lambda}, \lambda \in \Lambda$ ($\nu = 1, 2, \cdots, n$), we can find a point $x \in R$ for which $a_i(x) = a_i$ for every $\lambda \in \Lambda$.

In the sequel, we consider generalization of this theorem in the case where $S_i$ ($\lambda \in \Lambda$) are merely compact.

Theorem 2. If all $S_i$ ($\lambda \in \Lambda$) are compact and for any system of points $a_i \in S_i$ ($\lambda \in \Lambda$) subject to the condition (F), we can find a point $x \in R$ for which $a_i(x) \in \{a_i\}^-$ for every $\lambda \in \Lambda$, then the weak topology of $R$ is compact.

Proof. Let $K$ be a maximal system of sets of $R$ subject to the condition (I) $\bigcap_{\lambda \in \Lambda} a_i^{-1}(K) \neq \emptyset$ for every finite number of sets $K_{\nu} \in \mathcal{K}$ ($\nu = 1, 2, \cdots, n$). We see easily then that $A \cap K \neq \emptyset$ for all $K \in \mathcal{K}$ implies $A \in \mathcal{K}$, and $L, K \in \mathcal{K}$ implies $L \cap K \in \mathcal{K}$. For any $\lambda \in \Lambda$, we have obviously $\bigcap_{\nu \in \nu} a_i(K_{\nu}) \neq \emptyset$ for every finite number of sets $K_{\nu} \in \mathcal{K}$ ($\nu = 1, 2, \cdots, n$), and hence $\bigcap_{\nu \in \nu} a_i(K) = \emptyset$, because $S_i$ is compact by assumption. For a point $a_i \in \bigcap_{\nu \in \nu} a_i(K)$, we have

$$a_i^{-1}(U) \in \mathcal{K} \quad \text{for} \quad a_i \in U \in \mathcal{N}_i,$$

because for $a_i \in U \in \mathcal{N}_i$, $K \in \mathcal{K}$ we have obviously

$$a_i(K \cap a_i^{-1}(U)) = a_i(K) \cap U \neq \emptyset$$

which yields $K \cap a_i^{-1}(U) \neq \emptyset$. Therefore the system of points $a_i$ ($\lambda \in \Lambda$) satisfies the condition (F), and hence we can find a point $x \in R$ by assumption such that $a_i(x) \in \{a_i\}^-$ for every $\lambda \in \Lambda$. For such a point $x \in R$, we have obviously $a_i(x) \in \bigcap_{\nu \in \nu} a_i(K_{\nu})^-$, and consequently $a_i^{-1}(U) \in \mathcal{K}$ for $a_i(x) \in U \in \mathcal{N}_i$, as proved just above. Therefore we have
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\[ \bigcap_{\nu=1}^{n} a_{\nu}^{-1}(U_{\nu}) \subseteq R \] for \( a_{\nu}(x) \in U_{\nu} \in R_{\nu} \) \((\nu = 1, 2, \ldots, n)\).

As all \( \bigcap_{\nu=1}^{n} a_{\nu}^{-1}(U_{\nu}) \) for every finite number of sets \( U_{\nu} \in R_{\nu} \) constitute a neighbourhood system of the weak topology of \( R \), we conclude that \( x \in K^{-} \) for all \( K \in R \). For any system of closed sets \( R \) subject to the condition (I), we can find by the maximal theorem a maximal system \( R \) subject to the condition (I) such that \( R \supseteq R \), and for such \( R \) we have

\[ \bigcap_{K \in R} K^{-} \supseteq \bigcap_{K \in R} K^{-} = \phi, \]

as proved just above. Thus the weak topology of \( R \) is compact.

Let \( S \) be a topological space with topology \( \mathcal{T} \). For every point \( a \in S \), we define a closed set \( a^{*} \) as

\[ a^{*} = \bigcap_{U \in \mathcal{T}} U^{-}. \]

With this definition we have obviously: \( \{ a \}^{-} \subseteq a^{*} \), and \( b \in a^{*} \) implies \( a \in b^{*} \).

**Theorem 3.** If the weak topology of \( R \) by \( a_{\lambda} (\lambda \in A) \) is compact, then for any system of points \( a_{\lambda} \in S_{\lambda} (\lambda \in A) \) subject to the condition (F) we can find a point \( x \in R \) for which \( a_{\lambda}(x) \in a_{\lambda}^{*} \) for all \( \lambda \in A \).

**Proof.** For a system of points \( a_{\lambda} \in S_{\lambda} (\lambda \in A) \) subject to the condition (F), we have

\[ \bigcap_{\lambda \in A} \bigcap_{a_{\lambda} \in u \in R} a_{\lambda}^{-1}(U)^{-} = \phi, \]

because \( R \) is compact by assumption. For a point \( x \in R \) such that \( x \in a_{\lambda}^{-1}(U)^{-} \), we have \( a_{\lambda}(x) \in U^{-} \) for all \( a_{\lambda} \in U \in R_{\lambda} \), as \( a_{\lambda}^{-1}(U)^{-} \subseteq a_{\lambda}^{-1}(U)^{-} \) (cf. TLTS §16, Theorem 3), we have \( a_{\lambda}(x) \in U^{-} \) for all \( a_{\lambda} \in U \in R_{\lambda} \), and hence \( a_{\lambda}(x) \in a_{\lambda}^{*} \) for all \( \lambda \in A \).

Finally we consider the topologies of \( S \) for which \( \{ a \}^{-} = a^{*} \) for every point \( a \in S \). We can prove easily:

**Lemma.** \( \{ a \}^{-} \ni b \) implies always \( \{ b \}^{-} \ni a \), if and only if \( a \in U \in \mathcal{T} \) implies \( \{ a \}^{-} \subseteq U \).

If \( \{ a \}^{-} = a^{*} \) for every point \( a \in S \), then for any point \( b \in \{ a \}^{-} \) we can find \( U \in \mathcal{T} \) such that \( a \in U \) and \( b \in U^{-} \), and hence by Lemma \( \{ a \}^{-} \subseteq U \) and \( \{ b \}^{-} \subseteq U^{-} \). Thus we have

**Theorem 4.** We have \( \{ a \}^{-} = a^{*} \) for every point \( a \in S \), if and only if the partition space of \( S \) by the partition \( \{ a \}^{-} (a \in S) \) is a Hausdorff space.

**Remark 1.** The condition about point system in Theorem 2 is not necessary. Because, let \( \{ a, b \} \) be a topological space with the topology: \( \{ a, b \}, \{ a \}, \phi \). The point set \( \{ a \} \) is obviously compact by the relative topology, but \( a \in \{ b \}^{-} = \{ b \} \).

**Remark 2.** The condition in Theorem 3 is not sufficient. Because, let \( \{ 0, 1, 2, \ldots \} \) be a topological space with a neighbourhood system: \( \{ 0, 1, 2, \ldots \}, \{ n \} (n=1, 2, \ldots) \). This space is obviously compact. It is clear that a point set \( \{ 1, 2, \ldots \} \) is not compact by the relative topology but we have \( \{ 0 \} \in \{ 0, 1, 2, \ldots \} \) for every \( n=1, 2, \ldots \).