1. Introduction. Let $G$ be a convex domain of the euclidean $n+1$-space $R_{n+1}$, $(-\infty < t < +\infty, -\infty < x_i < +\infty \ (i=1, 2, \ldots, n))$, containing a curve $C: \{(t, x_i(t)) \mid t \in [a, b]\}$, where $x_i(t) \in C^2[a, b]$.

Consider real solutions $u$ of an inequality of the following kind:

$$\frac{\partial u(t, x)}{\partial t} - a_{ij}(t, x) \frac{\partial^2 u(t, x)}{\partial x_i \partial x_j} \leq M \left( \sum_{i=1}^{n} \left| \frac{\partial u(t, x)}{\partial x_i} \right| + |u(t, x)| \right).$$

Here $((a_{ij}(t, x)))$ denotes a positive definite, symmetric matrix of real valued functions $a_{ij}(t, x) \in C^2(G)$, and $M$ a constant.

Our purpose in this note is to prove the following theorem for solutions of (1.1).

Theorem. If $u$ is a solution of (1.1) in the convex domain $G$ and if for any $\alpha > 0$,

$$\lim_{\tau \to 0} \max_{|x-x(t)|=\tau} \left\{ \left| u(t, x) \right|, \left| \frac{\partial u(t, x)}{\partial t} \right|, \left| \frac{\partial u(t, x)}{\partial x_i} \right|, \left| \frac{\partial^2 u(t, x)}{\partial x_i \partial x_j} \right| \right\} \cdot |x-x(t)|^{-\alpha} = 0$$

then $u$ vanishes identically in the horizontal component.

The method is based upon the ideas of H. O. Cordes [2] and E. Heinz [3]. The tools used are all elementary, but our proof is somewhat complicated.

2. The Cordes' transformation. Assuming $[a, b] \supset [-\varepsilon, 1+\varepsilon]$ ($\varepsilon > 0$), let $\hat{A}(t)$ be the positive square root of the matrix $A(t) = ((a_{ij}(t, x(t))))$. Let

$$x-x(t)=\hat{A}(t)\bar{x} \quad \text{for} \quad t \in [-\varepsilon, 1+\varepsilon],$$

then we may assume that for some $R_1 > 0$,

a) $a_{ik}(t, \bar{x}) \in C^1([-\varepsilon, 1+\varepsilon] \times D_{R_1}) \ (D_{R_1} = \{x \mid |x| \leq R_1\})$,

b) $a_{ik}(t, 0) = \delta_{ik}$,

c) there are positive numbers $C_1$ and $C_2$ such that for any real vector $(\xi_1, \xi_2, \ldots, \xi_n)$

$$C_1 \sum_{i=1}^{n} \xi_i^2 \leq \sum_{i=1}^{n} a_{ij}(t, \bar{x}) \xi_i \xi_j \leq C_2 \sum_{i=1}^{n} \xi_i^2.$$

From (a), (b) and (c) we see the following

Lemma 1. For some $R_2, \tilde{R}_2 < R_1$ there is a topological transformation from $[-\varepsilon, 1+\varepsilon] \times D_{R_2}$ onto $[-\varepsilon, 1+\varepsilon] \times D_{\tilde{R}_2}$:

$$\tilde{y} = \tilde{y}(t, \bar{x}), \quad t = t$$

such that it satisfies the following conditions:
I. 1) \( \dot{y}(t, 0) = 0, \)
2) \( \frac{\partial \dot{y}_i}{\partial x_j}, \frac{\partial \dot{y}_i}{\partial y_j}, \frac{\partial ^2 \dot{y}_i}{\partial x_j \partial x_k}, \frac{\partial ^2 \dot{y}_i}{\partial x_j \partial y_k} \) are continuous over \([-\varepsilon, 1+\varepsilon] \times (D\bar{R}_2 - \{0\}) \)

\( \frac{\partial \dot{y}_i}{\partial x_j} \) is continuous over \([-\varepsilon, 1+\varepsilon] \times (D\bar{R}_2 - \{0\}) \) and \( \frac{\partial \dot{y}_i}{\partial t} \) is continuous over \([-\varepsilon, 1+\varepsilon] \times (D\bar{R}_2 - \{0\}) \).

II. for any \( \ddot{y} : 0 < |\ddot{y}| \leq \bar{R}_2 \), there is a suitable polar coordinates \((r, \varphi_e)\) such that

\[
(2.1) \quad \dot{\theta} a_i(t, \ddot{y}) = p(t, \ddot{y}) \left( \frac{\partial ^2 \theta}{\partial r^2} + \frac{n-1}{r} \frac{\partial \theta}{\partial r} + N^1 \right) \phi_i + p_i(t, \ddot{y}) \frac{\partial \phi_i}{\partial \ddot{y}},
\]

where \( p(t, \ddot{y}), p_i(t, \ddot{y}), \) and the operator \( N \) satisfy the following conditions:

1. \( C > p(t, \ddot{y}) > C', \quad |p_i(t, \ddot{y})| < C \),
2. \( |p(t, \ddot{y})| < C, \quad \left| \frac{\partial p(t, \ddot{y})}{\partial t} \right| < C, \quad \left| \frac{\partial p(t, \ddot{y})}{\partial \varphi_e} \right| < C \),
3. \( \frac{\partial a_i}{\partial \varphi_e} = \frac{\partial a_i}{\partial \varphi_e} = \frac{\partial a_i}{\partial \varphi_e} = \frac{\partial a_i}{\partial \varphi_e} \), \( \lambda(y) = \frac{d\Omega}{d\varphi_1, d\varphi_2, \ldots, d\varphi_{n-1}} \),

where \( d\Omega \) is the usual surface element of the unit sphere,

4. there are two positive numbers \( C_1, C_2 \) such that

\[
C_1 \sum_{i=1}^{n-1} \gamma_i^2 \leq \sum \bar{a}_{sr}(t, \ddot{y}) \gamma_{rs} = C_2 \sum_{i=1}^{n-1} \gamma_i^2,
\]

for any real vector \( \{\gamma_1, \ldots, \gamma_{n-1}\} \),

5. \( \bar{a}_{sr}, \frac{\partial \bar{a}_{sr}}{\partial t}, \frac{\partial \bar{a}_{sr}}{\partial r} \) and \( \frac{\partial \bar{a}_{sr}}{\partial \varphi_e} \) are continuous and

\[
|\bar{a}_{sr}| < C, \quad \left| \frac{\partial \bar{a}_{sr}}{\partial t} \right| < C, \quad \left| \frac{\partial \bar{a}_{sr}}{\partial r} \right| < C, \quad \left| \frac{\partial \bar{a}_{sr}}{\partial \varphi_e} \right| < C,
\]

where the constants \( C_1, C_2 \) and \( C \) depend only on \( R_1, C_1, C_2 \) and the derivatives of \( a_{ij}(t, x) \) of order \( \leq 2 \). (Here we use a finite number of fixed, suitable systems of polar-coordinates covering the unit sphere.)

To prove the above proposition, we only remark that

\[
\nu_c(t, r, \theta, \theta_2, \ldots, \theta_{n-1}) = \sum a_{ik}(t, \ddot{x}) \frac{\ddot{x}_k}{r^2} \frac{\partial x_k}{r},
\]

satisfies the following conditions: for any \( \ddot{x} \in [-\varepsilon, 1+\varepsilon] \) and \( \ddot{x} : 0 \leq |\ddot{x}| < R_1 \) the function \( \nu_c(t, r, \theta), \nu_{\|\ddot{x}_e\parallel}, \nu_{\|\ddot{x}_2\parallel}, \nu_{\|\ddot{x}_3\parallel}, \nu_{\|\ddot{x}_e\parallel}, \nu_{\|\ddot{x}_2\parallel}, \nu_{\|\ddot{x}_3\parallel}, \nu_{\|\ddot{x}_e\parallel}, \) and \( \nu_{\|\ddot{x}_e\parallel}, \nu_{\|\ddot{x}_2\parallel}, \nu_{\|\ddot{x}_3\parallel} \) are all continuous, and for any \( t \in [-\varepsilon, 1+\varepsilon] \) and \( \ddot{x} : 0 < |\ddot{x}| \leq R_1 \) \( \nu_{\|\ddot{x}_e\parallel}, \nu_{\|\ddot{x}_2\parallel}, \nu_{\|\ddot{x}_3\parallel} \) and \( \nu_{\|\ddot{x}_e\parallel} \) are continuous, where \( \nu(t, r, \theta) \) is considered as a function of \( t, r, \theta \). Here and in the proof of the following sections \( \nabla_{\partial h} \) denotes \( \frac{\partial u}{\partial h} \).
Furthermore by the transformation: \((t, \bar{y}) \rightarrow (t, y, \varphi) \rightarrow (t, s, \varphi) = (t, y)\):
\[
e(t, y) = re^{\int^r_{r_0} e^{-m_0 s - \int^s_{r_0} \frac{dr}{r}}} \]
we see the following

**Lemma 2.** By the transformation \((t, \bar{y}) \rightarrow (t, y)\) with a sufficiently large \(m_0\), the following condition is satisfied: for any \(w \in C^2(y : |y| = 1)\) and for any \(t \in [-\varepsilon, 1+\varepsilon]\),

\[
III. \frac{\partial}{\partial s} \int Nw \cdot \omega dO_1 \leq m_0 \int Nw \cdot \omega dO_1 < 0
\]
as well as Conditions I and II.

3. The first inequality. Using the above lemmas, we will deduce the Heinz' inequality with respect to (1.1). For this purpose we may assume that

\[
L_1(u) = q(t, x) \frac{\partial u}{\partial t} - a_{ij}(t, x) \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i(t, x) \frac{\partial u}{\partial x_i}
\]
(3.1)

where \(q(t, x)(>a>0) \in C^1(t, \varphi, \alpha)\), \(a_{ij}(t, x) \in C^0(t, x)\), \(b_i(t, x) \in \bar{C}^0(t, x)\) and the coefficients of \(N \in \tilde{C}^1(t, r, \varphi)\) \((0<r<R)\) for fixed, suitable polar coordinates \((r, \varphi)\) of \(x\).

Furthermore we may assume that \(u\) satisfies the condition (1.2) with \(x(t) = 0\) \(\text{for } t \in [-\varepsilon, 1+\varepsilon]\).

Put \(D_{r_0, K_0} = \{(t, x) | 0 \leq t \leq 1 \text{ and } |x| \leq r_0 \wedge K_0^{-1}t\}\) and let \(\varphi_{r_0, K_0}(t, x)\) be such that: (1) it is in \(C^2(D_{r_0, K_0} - \{0\})\), (2) its carrier is contained in \(D_{r_0, K_0}\), (3) \(\varphi_{r_0, K_0} = 1\) in \(D_{\frac{r_0}{2}, \frac{K_0}{2}} - \{0\}\) and (4) \(\psi = u \psi_{r_0, K_0}\) also satisfies the condition (1.2).

Furthermore let \(f\) be a monotone decreasing, smooth function such that

\[
f(t) = 1 \text{ for } t \leq \frac{2}{3}, \quad f(t) > 0 \text{ for } t < 1 \text{ and } f(1) = 0.
\]

Let \(\alpha(t) = \alpha f(t) + (n-2)\). Finally let \(\varphi(t)\) be a monotone increasing, smooth function such that

\[
\varphi(t) = t \text{ for } t \leq \frac{1}{4}, \quad \varphi(t) = 1 \text{ for } t \geq \frac{1}{2}
\]
and let \(\Phi(t) = \varphi(t)^{\alpha_0} \varphi^t\). Then we see the following

**Lemma 3.** For sufficiently small \(r_0\) and sufficiently large \(K_0\) and \(k\) there is a constant \(\alpha_0\) such that for any \(\alpha > \alpha_0\),

\[
\alpha^2 k K_1 \left[ \int_{D_{r_0, K_0}} |v|^{2-r-a(t)} \Phi(t) dx \right] dt
\]
(3.2)

\[
\leq \int_{D_{r_0, K_0}} \left[ L_1(v)^{2-r-a(t)} \Phi(t) dx + \alpha^2 K_2 \int_{D_{r_0}} |v|^{2-r-a(t)} \Phi(t) dx \right] dt - 1.
\]
where $K_0, K_1, K_2$ are constant numbers depending only the derivatives with respect to $t, r, \gamma$, of $q$ of order $\leq 1$, the derivatives with respect to $t, r$ of the coefficients of $N$, and $f$ of order $\leq 1$, which are independent of systems of polar-coordinates $\{r, \gamma\}$. (Here and in the following proofs we denote such constants by $K$.)

(Outline of the proof). By the usual limit processes [1, 2] we may assume that the coefficients of $L_1$ and $v$ are sufficiently smooth. Let $\beta(t) = \frac{1}{2}(\alpha(t) - n + 2)$ and $u = r^{\beta(t)} \gamma$. Then we see that

$$\int \int |L_1(v)|^2 r^{2-\beta(t)} \Phi_a(t) dx \, dt$$

(3.3)

$$\geq \int \int \left( |qrz|_{x^2}^2 + |L^* z|_{x^2}^2 + 2L^* z \cdot L^* z - 2r^2 z_{x^2} \cdot q \cdot (L^* z + L^* z) \right) \cdot \Phi_a(t) dO, dr, dt,$$

where

$$L^* z = r(rz + rz) IR + N z + \frac{a^2 - (n - 2)^2}{4} - q \alpha f'(t) r^2 \log r \right] z,$$

$$L^* z = \alpha rz,$$

From $\phi' \geq 0$ it implies that for any $K$ there is a number $k_0$ such that for $k > k_0$

$$(q \Phi_a)_t - K(q \Phi_a) \geq \frac{1}{2} (q \Phi_a)_t.$$

Therefore by partial integrations with respect to $t$ and $r$ and from III in § 2 and the relation $f' \leq 0$, it follows that

$$(3.3) \geq \int \int \left( r^2 q^2 \Phi_a(z_{x^2})^2 + a^2 r \Phi_a(z_{x^2})^2 - 2(a - 2) r^2 q \Phi_a(z_{x^2}), z_{x^2} \right.$$

$\left. + 2r^2 (q \Phi_a)_t z_{x^2} - q \Phi_a(z_{x^2})^2 - K^2 r^2 (q \Phi_a)_t z_{x^2}^2 \right) dO, dr, dt$$

$$- a^2 K^2 \int \int r \Phi_a(z_{x^2})^2 dO, dr, dt$$

$$+ \int \int \left\{ -a m \Phi_a \cdot M + r(q \Phi_a) z_{x^2} - 2r(q \Phi_a) z_{x^2} - r^2 (q \Phi_a)_t z_{x^2} \right\} dO, dr, dt$$

$$+ \int \int \left\{ q r \Phi_a \cdot z \cdot (r z_{x^2}) + 2r(q \Phi_a) z_{x^2} - 2r^2 (q \Phi_a)_t z_{x^2} \right\} dO, dr, dt$$

$$+ (q \Phi_a)_t r^2 z_{x^2} - 2r (q \Phi_a)_t z_{x^2} + M z + r \Phi_a \cdot \frac{a^2 - (n - 2)^2}{4} z_{x^2}$$

$$- a r^2 (q \Phi_a) \cdot K z_{x^2}^2 \right) dO, dr \bigg|_{x=1}.$$

Furthermore we note that for sufficiently small $r_0$, for sufficiently large $K_0$ and $h$, there is a number $\alpha_0$ such that for any $\alpha > \alpha_0$

$$(3.4)_1 \quad \alpha \Phi_a - r^2 (q \Phi_a)_t \geq 0,$$

$$(3.4)_2 \quad m \alpha \Phi_a - r (q \Phi_a)_t + rq \Phi_K \geq 0.$$

From (3.4)_1, (3.4)_2 and II in § 2, it follows that

$$(3.3) \geq K_0 \alpha^2 k \int \int r z^2 \Phi_a dO, dr dt - K_2 \alpha^2 \int \int r z^2 \Phi_a dO, dr \bigg|_{x=1},$$
which implies (3.1).

4. The second inequality. Let $r_0$ and $K_0$ be fixed numbers such that for sufficiently large $k$ and $\alpha_0$, (3.2) is valid.

Then using the relation: $f(1) = 0$, $f(t) > 0$ for $t < 1$ and $\varphi(t) = 1$ for $t \geq \frac{1}{2}$,

we see that even if $\int \frac{\partial}{\partial x} f(t) \left| v \right|^2 dx \bigg|_{t=1} = 0$, there is an interval $[c, d]$ ($\frac{1}{2} < c < d < 1$) such that for any $k$ and for any $\alpha (\alpha > \alpha_0(k, u))$

$$\int \left| v \right|^2 r^{-\alpha(t)} \Phi_n(t) dx \bigg|_{t=1} \leq \int \left| v \right|^2 r^{-\alpha(t)} \Phi_n(t) dx \bigg|_{t \in [c, d]}.$$

Therefore from (3.2) it follows that for sufficiently large $k$ there is a constant $K_8$ and $\alpha_0$ such that for $\alpha > \alpha_0$

$$\alpha^2 K_8 \int \left| v \right|^2 r^{-\alpha(t)} \Phi_n(t) dx \leq \int \left| L_1(v) \right|^2 r^{-\alpha(t)} \Phi_n(t) dx dt.$$

Then from (3.1), (4.1) and (3.4), we see the following

**Lemma 4.** For sufficiently small $r_0$ and for sufficiently large $K_0$ and $k > k_0$, there are constants $K_9$ and $\alpha_0$ such that for $\alpha > \alpha_0$

$$\alpha^2 K_9 \int \left| v \right|^2 r^{-\alpha(t)} \Phi_n(t) dx \leq \int \left| L_1(v) \right|^2 r^{-\alpha(t)} \Phi_n(t) dx dt,$$

where $k_0$ depends on $u$ and $K_0$.

5. The proof of Theorem. In this section we use the notations in §1 and §2. By (1.1) we may assume that for some $r_0, \varepsilon$

$$\left| L_1(u) \right| \leq M \left\{ \left| u \right| + \sum_{i=1}^n \frac{\partial u}{\partial y_i} \right\} \text{ for } t \in [-\varepsilon, 1 + \varepsilon] \text{ and } r \leq r_0$$

where $2K_0 r_0 < \frac{1}{4}$.

Then from Lemma 4 we see that for any $\alpha (\alpha > \alpha_0(K, r_0, k))$

$$\int \int_{D_{r_0/2, 2K_0}} \left( \left| u \right|^2 + \left| u_{y_1} \right|^2 \right) r^{-\alpha(t)} \Phi_n dx dy dt$$

$$\leq k^{-\frac{1}{2}} K \int \int_{D_{r_0, K_0}} \left| L_1(v) \right|^2 r^{-\alpha(t)} \Phi_n dx dy dt$$

$$\leq k^{-\frac{1}{2}} K \int \int_{D_{r_0, K_0} - D_{r_0/2, 2K_0}} \left| L_1(v) \right|^2 r^{-\alpha(t)} \Phi_n dx dy dt$$

$$+ k^{-\frac{1}{2}} K \cdot M \int \int_{D_{r_0/2, 2K_0}} \left( \left| u \right|^2 + \left| u_{y_1} \right|^2 \right)^2 r^{-\alpha(t)} \Phi_n dx dy dt.$$

Accordingly choosing $k$ sufficiently large such that $2K \cdot M < k^{-\frac{1}{2}}$,

it follows that for any $\alpha > \alpha_0$
Therefore tending $\alpha \to \infty$, we see that
\[ u(t, y) = 0 \quad \text{for} \quad t \in \left[ \frac{1}{4}, \frac{2}{3} \right], \quad r \leq r_0/3K_0. \]

Since, in the above proof, the numbers $\{\varepsilon, \frac{1}{4}\}$ and $\frac{2}{3}$ may be replaced by arbitrary small and large numbers respectively, we see that $u(t, x) = 0$ in a neighbourhood of $C$ in $(a, b) \times R_x$. Then by a topological argument and from Lemma 1 and Lemma 4 also, we see that $u(t, x) = 0$ in the horizontal component stated in § 1.

Another detailed proof of Theorem and the results in my previous paper [4] with other consequences will be published in the Osaka Mathematical Journal next year.

References


