13. On Rings of Continuous Functions and the Dimension of Metric Spaces

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M. Katětov [1] has once established an interesting theory on a relation between the inductive (Menger-Urysohn) dimension of a compact space R and the structure of the ring of all continuous functions on R . The purpose of this brief note is to give a slight extension to Katětov's theory for a metric space while simplifying his discussion.

According to [1], we consider an *analytical ring*, i.e. a commutative topological ring with a unit e and a continuous real scalar multiplication. A subring C_1 of an analytical ring C is called *analytically closed* if

(1) $\lambda e \in C_1$ for any real λ , (2) $x \in C_1$ whenever $x \in C$, $x^n + a_1 x^{n-1} + \dots + a_n = 0$, $a_i \in C_1$, (3) $\bar{C}_1 = C_1$.

Let C' be a subset of C ; then a subset M of C is called an *analytical base* of C' in C if there exists no analytically closed subring $C_1 \not\supset C'$ containing M . The least number of an analytical base of C' in C is called the *analytical dimension* of C' in C and denoted by $\dim(C', C)$. The ring $C(R)$ of all bounded real-valued continuous functions of R is an analytical ring as for its strong topology. We denote by $U(R)$ the subset of $C(R)$ consisting of all uniformly continuous functions. Furthermore, according to [2], we call a continuous mapping f of a metric space R into a metric space S *uniformly 0-dimensional* if for any $\varepsilon > 0$ there exists $\eta > 0$ such that $\delta(U) < \varepsilon$ whenever $U \subset R$, $\text{diam } f(U) < \eta$, where $\delta(U) < \varepsilon$ means the fact that there exists an open covering $\mathfrak{B}$ of U such that $\text{mesh } \mathfrak{B} = \sup \{\text{diam } V \mid V \in \mathfrak{B}\} < \varepsilon$ and $\text{order } \mathfrak{B} \leq 1$. The covering dimension of R or the strong inductive dimension of R as the same is denoted by $\dim R$. Now we can prove the following

Theorem. $\dim R = \dim(U(R), C(R))$ for every locally compact, metric space R .

To establish this theorem we prove some lemmas.

Lemma 1. Let $f(x) = (f_1(x), \dots, f_n(x))$ be a uniformly 0-dimensional, bounded mapping of a metric space R into the n -dimensional Euclidean space E_n . Let C_1 be an analytically closed subring of $C(R)$ containing $f_1, \dots, f_n$; then for every sets F and G of R with distance $(F, G) = d(F, G) > 0$, there exists $g \in C_1$ such that $g(F) \geq 1$, $g(G) = 0$, where $g(F) \geq 1$, for example, means that $g(x) \geq 1$ for every $x \in F$.

Proof. Let $d(F, G) = \varepsilon > 0$ and take $\eta > 0$ such that $\text{diam } f(U) < \eta$ for $U \subset R$ implies $\delta(U) < \varepsilon$. Choosing $\xi > 0$ such that $\text{diam } \prod_{i=1}^n (r_i - 2\xi, r_i + 2\xi) < \eta$ for every r_i , we cover $f(R)$ with finitely many cubes

$$I_k = \prod_{i=1}^n [r_{ki} - \xi, r_{ki} + \xi], \quad k=1, \dots, l.$$

Let

$$U_k = f^{-1}(I_k), \quad V_k = f^{-1}(J_k),$$

where

$$J_k = \prod_{i=1}^n (r_{ki} - 2\xi, r_{ki} + 2\xi).$$

It easily follows from $f_i \in C_1$ that $f_{ki} = (2\xi - |f_i - r_{ki}|) \frac{1}{\xi} \in C_1$, and hence

$$\bar{f}_k = \prod_{i=1}^n f_{ki} \in C_1. \quad \text{Then}$$

$$\bar{f}_k(x) \geq 1 \quad \text{for every } x \in U_k,$$

$$\bar{f}_k(x) > 0 \quad \text{for every } x \in V_k,$$

$$\bar{f}_k(x) = 0 \quad \text{for every } x \in \bar{V}_k - V_k.$$

Since $\text{diam } f(V_k) = \text{diam } J_k < \eta$, we can find an open covering $\mathfrak{B}_k$ of V_k with mesh $\mathfrak{B}_k < \varepsilon$, order $\mathfrak{B}_k \leq 1$. It is easy to see that $S(F, \mathfrak{B}_k) = W_k$ is an open closed set of V_k satisfying $W_k \cap G = \emptyset$, $W_k \supset F \cap V_k$. Now we define a function g_k by

$$g_k(x) = \bar{f}_k(x), \quad x \in W_k,$$

$$g_k(x) = 0, \quad x \notin W_k.$$

Then since g_k clearly satisfies $g_k \in C(R)$ and $g_k^2 - \bar{f}_k g_k = 0$ for $\bar{f}_k \in C_1$, we get $g_k \in C_1$ satisfying $g_k(F \cap U_k) \geq 1$, $g_k(G) = 0$, $g_k \geq 0$. Letting $g = \sum_{k=1}^l g_k$ we have an element g of C_1 satisfying $g(F) \geq 1$, $g(G) = 0$, $g \geq 0$.

Lemma 2. $\dim R \geq \dim(U(R), C(R))$ for every metric space R .

Proof. If $\dim R \leq n$, then by [2] there exists a uniformly 0-dimensional, bounded mapping $f(x) = (f_1(x), \dots, f_n(x))$ of R into E_n . Hence any analytically closed subring C_1 of $C(R)$ containing $f_1, \dots, f_n$ also contains, for every disjoint closed sets F and G with $d(F, G) > 0$, $\varphi \in C_1$ such that $\varphi(F) = 0$, $\varphi(G) \geq 1$ by Lemma 2. Hence by an analogous theorem to that of E. Hewitt [3, Theorem 1], we get, for every $\bar{\varphi} \in U(R)$ and $\varepsilon > 0$ a polynomial $P(\varphi_1, \dots, \varphi_k)$ in $\varphi_i \in C_1$, $i=1, \dots, k$ such that $|\bar{\varphi} - P(\varphi_1, \dots, \varphi_k)| < \varepsilon$. Therefore $\bar{\varphi} \in \bar{C}_1 = C_1$, which implies $C_1 \supseteq U(R)$. Thus $(f_1, \dots, f_n)$ is an analytical base of $U(R)$ in $C(R)$, i.e. $\dim(U(R), C(R)) \leq n$.

Lemma 3. $\dim R \leq \dim(U(R), C(R))$ for every locally compact, metric space R .

Proof. Let $(f_1, \dots, f_n)$ be an analytical base of $U(R)$ in $C(R)$; then $f(x) = (f_1(x), \dots, f_n(x))$ is a bounded continuous mapping of R onto a subset $f(R)$ of E_n . Since R is locally compact, there is a locally finite closed covering $\{R_\alpha \mid \alpha \in \Omega\}$ consisting of compact sets R_α . Let $\mathfrak{U}$ be any finite open covering of R_α ; then there exists, for every $q \in f(R)$,

a nbd (=neighborhood) $V(q)$ of q in $f(R)$ such that $\delta(f^{-1}V(q)) \leq \mathfrak{U}$, i.e. there exists an open covering $\mathfrak{B}$ of $f^{-1}V(q)$ satisfying $\mathfrak{B} < \mathfrak{U}$ in R_α and order $\mathfrak{B} \leq 1$. It is enough to prove this proposition just for every binary open covering $\mathfrak{U}$ of R_α . For we can find, for every finite open covering $\mathfrak{U}$ of R_α , binary open coverings $\mathfrak{U}_1, \dots, \mathfrak{U}_k$ of R_α satisfying $\mathfrak{U}_1 \wedge \dots \wedge \mathfrak{U}_k < \mathfrak{U}$. Then $\delta(f^{-1}V_i(q)) \leq \mathfrak{U}_i$, $i=1, \dots, k$ for nbds $V_i(q)$, $i=1, \dots, k$ of q imply $\delta(f^{-1} \bigcap_{i=1}^k V_i(q)) \leq \mathfrak{U}$. Now assume the contrary, i.e. let F and G be disjoint closed sets of R_α such that $\delta(f^{-1}V(q)) \not\leq \{F^c, G^c\}$ for every nbd $V(q)$ of q .

Let $D = \{g \mid g \in C(R), \text{ for every } \varepsilon > 0, \text{ there exist a nbd } V(q) \text{ of } q \text{ in } f(R) \text{ and an open covering } \mathfrak{U} \text{ of } f^{-1}V(q) \text{ such that } \text{mesh } g(\mathfrak{U}) < \varepsilon \text{ and order } \mathfrak{U} \leq 1\}$, where $g(\mathfrak{U})$ denotes the covering $\{g(U) \mid U \in \mathfrak{U}\}$ then D is an analytically closed subring containing $f_1, \dots, f_n$. Let us just show that $g \in D$ whenever $g \in C(R)$, $g^n + a_1 g^{n-1} + \dots + a_n = 0$, $a_i \in D$, where this n is not related with the number of f_i . Let us denote by $g_k(b_1, \dots, b_n)$, $k=1, 2, \dots, n$ the n roots of the equation

$$y^n + b_1 y^{n-1} + \dots + b_n = 0.$$

Let $|a_i| \leq K$, $i=1, \dots, n$; then since $g_k(b_1, \dots, b_n)$ are continuous functions of $b_1, \dots, b_n$ and accordingly are uniformly continuous for $|b_i| \leq K$, $i=1, \dots, n$, for any $\varepsilon > 0$ we can find $\delta > 0$ such that

$$|b_i - b'_i| < \delta, |b_i| \leq K, |b'_i| \leq K, i=1, \dots, n \text{ imply}$$

$$|g_k(b_1, \dots, b_n) - g_k(b'_1, \dots, b'_n)| < \frac{\varepsilon}{n}, k=1, \dots, n.$$

Now let $V(q)$ be a nbd of q and $\mathfrak{U} = \{U_\gamma \mid \gamma \in \Gamma\}$ an open covering of $f^{-1}V(q)$ such that $\text{mesh } a_i(\mathfrak{U}) < \delta$, $i=1, \dots, n$ and order $\mathfrak{U} \leq 1$. Moreover, let

$$\begin{aligned} \{x \mid g_k(a_1(x), \dots, a_n(x)) - g(x) = 0, x \in U_\gamma\} &= U_{k\gamma}, \\ \{U_{k\gamma} \mid k=1, \dots, n\} &= \mathfrak{U}_\gamma, \\ \{S^n(U_{k\gamma}, \mathfrak{U}_\gamma) \mid U_{k\gamma} \in \mathfrak{U}_\gamma\} &= \mathfrak{B}_\gamma. \end{aligned}$$

Then $\mathfrak{B}_\gamma$ is an open covering of U_γ with order $\mathfrak{B}_\gamma \leq 1$ and $\text{mesh } g(\mathfrak{B}_\gamma) < \varepsilon$; hence $\mathfrak{B} = \bigcup \{\mathfrak{B}_\gamma \mid \gamma \in \Gamma\}$ is an open covering of $f^{-1}V(q)$ with order $\mathfrak{B} \leq 1$ and $\text{mesh } g(\mathfrak{B}) < \varepsilon$. Thus we get $g \in D$. Since R_α is compact, it must be $d(F, G) > 0$, and hence there exists a function $h \in U(R)$ such that $h(F) = 0$, $h(G) = 1$. However, from the assumption D does not contain such a function h , which contradicts the fact that $(f_1, \dots, f_n)$ is an analytical base of $U(R)$ in $C(R)$. Hence for every finite open covering $\mathfrak{U}$ of R_α and for every point q of $f(R)$ there exists a nbd $V(q)$ of q satisfying $\delta(f^{-1}V(q)) \leq \mathfrak{U}$. Take an open refinement $\mathfrak{B} = \{V_\gamma \mid \gamma \in \Gamma\}$ of $\{V(q) \mid q \in f(R)\}$ with order $\mathfrak{B} \leq n+1$. Then since $\delta(f^{-1}(V_\gamma)) \leq \mathfrak{B}$, we can find an open covering $\mathfrak{B}_\gamma$ of $f^{-1}(V_\gamma)$ satisfying $\mathfrak{B}_\gamma < \mathfrak{U}$, order $\mathfrak{B}_\gamma \leq 1$. Now $\mathfrak{B} = \bigcup \{\mathfrak{B}_\gamma \mid \gamma \in \Gamma\}$ restricted in R_α is an open refinement of $\mathfrak{U}$ with order $\mathfrak{B} \leq n+1$. Therefore we can conclude $\dim R_\alpha \leq n$. Hence, by

use of the sum-theorem, we get $\dim R \leq n$.

Combining Lemma 3 with Lemma 2, we can conclude the validity of the theorem.

Incidentally, let us show the following

Corollary. *$U(R)$ of every metric space R has an analytical base in $C(R)$ consisting of countably many elements.*

While checking up the proofs of Lemmas 1, 2, we know that this corollary is a direct consequence of the following

Lemma 4. *Every metric space R can be mapped into the Hilbert cube I_ω by a uniformly 0-dimensional mapping.*

Proof. Since, by [4], every metric space R can be imbedded in a product of countably many one-dimensional metric spaces, we can conclude this lemma from the fact owing to [2] that every one-dimensional metric space is mapped into E_1 by a uniformly 0-dimensional, bounded function.

References

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