20. The Thickening of Combinatorial n-manifolds in (n + 1)-space

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The sets which come into consideration are all to be polyhedral in some Euclidean space and manifolds, cells, spheres are to be combinatorial; all homeomorphisms, imbeddings are to be piecewise linear.

The regular neighborhood is originally defined by J. H. C. Whitehead, which is not necessary the neighborhood in the set theoretic sense. We put some restrictions to it as follows.

Definition. Let \( P \) be a finite polyhedron imbedded in an \( m \)-manifold \( W \) without boundary. The regular neighborhood \( U(P, W) \) of \( P \) in \( W \) means an \( m \)-manifold contained in \( W \) and containing \( P \) in the interior, which contracts geometrically into \( P \).

Then the results of Whitehead imply the following

Theorem 1. Let \( P \) be a finite polyhedron imbedded in a manifold \( W \) without boundary. Then for any two regular neighborhoods \( U_1(P, W) \) and \( U_2(P, W) \) of \( P \) in \( W \) there is a homeomorphism onto \( \psi : W \to W \) such that \( \psi(U_1(P, W)) = U_2(P, W) \) and \( \psi | P = \text{identity} \) where \( \psi \) is an orientation preserving homeomorphism if \( W \) is orientable.

The combinatorial version of the Schönflies conjecture for dimension \( n \) is the following statement: Let an \((n-1)\)-sphere \( S_{n-1} \) be imbedded in Euclidean \( n \)-space \( R^n \). Then the closure of the bounded component of \( R^n - S_{n-1} \) is an \( n \)-cell.

This has been affirmatively proved for \( n \leq 3 \). Theorem 1 enables us to prove the following

Theorem 2. Let a compact, \( n \)-manifold \( M_i \) without boundary be imbedded into an orientable, oriented \((n+1)\)-manifold \( W_i \) without boundary, \( i = 1, 2 \). Let \( U(M_i, W_i) \) be a regular neighborhood of \( M_i \) in \( W_i \) and \( \phi : M_1 \to M_2 \) be a homeomorphism onto.

Suppose that the combinatorial version of the Schönflies conjecture is true for dimension \( \leq n \).

Then there is a homeomorphism onto \( \psi : U(M_1, W_1) \to U(M_2, W_2) \) such that \( \psi | M_i = \phi \) and such that the oriented image of oriented

$U(M_1, W_1)$ is the oriented $U(M_2, W_2)$ where the orientation of $U(M_1, W_1)$ is induced by that of $W_i$.

In the proof of Theorem 2 we make extensive use of combinatorial methods and results of V. K. A. M. Gugenheim.\(^3\)

As consequents of Theorem 2 we have the following theorems.

Theorem 3. Let a compact, orientable $n$-manifold $M$ without boundary be imbedded in an orientable $(n+1)$-manifold $W$ without boundary. Let $U(M, W)$ be a regular neighborhood of $M$ in $W$.

Suppose that the combinatorial version of the Schönflies conjecture is true for dimension $\leq n$.

Then there is a homeomorphism into $\theta : M \times J \to W$ such that $\theta(x, 0) = x$ for all $x \in M$ and such that $\theta(M \times J) = U(M, W)$, where $J$ is the interval $-1 \leq s \leq 1$.

Theorem 4. Let $M$ be a compact, orientable $n$-manifold without boundary imbedded in an orientable $(n+1)$-manifold $W$ without boundary. Let $\phi : M \to M$ be a homeomorphism which is onto isotopic to the identity.

Suppose that the combinatorial version of the Schönflies conjecture is true for dimension $\leq n$.

Then there is an orientation preserving homeomorphism onto $\psi : W \to W$ such that $\psi | M = \phi$.

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