7. On Transformation of the Seifert Invariants

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The theory of continuous transformations of manifolds shows preference to the case that \( \dim X = \dim Y \) or \( \dim X > \dim Y \) where \( X \) is mapped into \( Y \). The reason is that every continuous mapping of an \( m \)-sphere into an \( n \)-sphere with \( m < n \) is homotopic to zero. We will cast a look on the case \( \dim X < \dim Y \).

1. Suppose \( z, z' \) are two disjoint zero-divisors in the compact manifold \( X \) such that \( \dim z + \dim z' \geq (\dim X) - 1 \). Then the pair \((z, z')\) determines [1] a rational interlacing cycle, \( \sigma(z, z') \), as follows. Let \( a, b \) be the smallest positive integers satisfying \( az \sim 0 \) and \( bz' \sim 0 \), and let \( A, B \) be two finite integral chains in \( X \) such that \( \partial A = az \) and \( \partial B = bz' \). Then, if \( f \) denotes the usual intersection function,

\[
\frac{1}{a} f(A, z') = \frac{1}{ab} f(A, \partial B) = \pm \frac{1}{ab} f(\partial A, B) = \pm \frac{1}{b} f(az, B) = \pm \frac{1}{b} f(z, B).
\]

One thus obtains an expression that does not depend on \( A \). Now

\[
\sigma(z, z') = \frac{1}{a} f(A, z')
\]

is Seifert’s interlacing cycle.

2. Let \( 2 \leq m < n \) be integers, let \( M \) be an \( m \)-dimensional and \( N \) an \( n \)-dimensional oriented differentiable compact manifold, moreover \( f : M \to N \) a continuous mapping. Let \( P, Q, R, S \) be pairwise disjoint oriented differentiable compact manifolds in \( N \) such that

\[
\begin{align*}
p &\geq n - m, & q &\geq n - m, & r &\geq n - m, & s &\geq n - m, \\
p + q + r + s &\leq 4n - m - 3, & p + q &\geq 2n - m,
\end{align*}
\]

where \( p, q, r, s \) are the dimensions of \( P, Q, R, S \) respectively. For instance setting

\[
p = q = r = n - 1 \quad \text{and} \quad s = n - m,
\]

one confirms at once that the above dimensional suppositions are fulfilled.

The algebraic inverse of \( P, Q, R, S \) under \( f \), defined for instance in [4], will be denoted by \( z_p, z_q, z_r, z_s \) respectively. Geometrically one can suppose [5] that the inverses of \( P, Q, R, S \) are differentiable manifolds. Then \( z_p, z_q, z_r, z_s \) is an integral cycle of dimension \( p - (n - m), q - (n - m), r - (n - m), \) and \( s - (n - m) \) respectively. Let the manifolds \( P, Q, R, S \) be defined in such a way that \( z_p, z_q, z_r, z_s \) are zero-divisors. That is always possible as one easily confirms. Let \( z_T \) denote the above defined Seifert interlacing cycle, \( \sigma(z_p, z_q) \). By

\[
\dim z_T = (\dim z_p) + 1 + \dim z_q - \dim M
\]

\[
= (p - n + m) + 1 + (q - n + m) - m = p + q - 2n + m + 1
\]
and the supposition \( p + q \geq 2n - m \), it follows that \( \dim z_T \geq 1 \).

Let \( a, b, c \) be the smallest positive integers such that \( cz_T \) is an integral cycle and that moreover
\[
az_S \sim 0 \quad \text{and} \quad bz_S \sim 0.
\]
Let \( A, B \) be chains in \( M \) satisfying \( \partial A = az_S \) and \( \partial B = bz_S \). Furthermore let \( Z_1, Z_2, \ldots \) be a base of the integral \((r+1)\)-cycles in \( M \) and \( Z'_1, Z'_2, \ldots \) be a base of the integral \((s+1)\)-cycles in \( M \). Now \( f \) being as above the intersection function, we set
\[
\zeta_i = f(A + Z_i, cz_T), \quad \zeta_{ij} = f(\zeta_{ti}, B + Z'_j).
\]
Then
\[
\dim \zeta_{ij} = \dim \zeta_i + (\dim z_S) + 1 - \dim M
\]
\[
= (\dim z_S) + 1 + \dim z_T - \dim M + (\dim z) + 1 - \dim M
\]
\[
= (\dim z_S) + 1 + (\dim z_T) + 1 + \dim z_T - \dim M - \dim M
\]
\[
= \dim z_T + \dim z_S + 3 - 3 \dim M
\]
\[
= p + q + r + s - 4n - m + 3 - 3m = (4n - m - 3) - 4n + m + 3 = 0.
\]
Thus the \( \zeta_{ij} \) are integers. The matrix consisting of these numbers is invariant under deformation of \( f \). In order that \( f \) is an essential map, it suffices that at least one \( \zeta_{ij} \) is not zero. To the matrix \( (\zeta_{ij}) \) there corresponds a comatrix that one obtains by projecting our results in the cohomology rings of \( M \) and \( N \), see for instance [2, 3].

3. Let \( r \) be a positive integer \( \leq m - 1 \) such that every integral homology class of dimension \( n-r-1 \) and likewise every such class of dimension \( n-m+r \) of \( N \) permits a realization 3 by an oriented differentiable compact manifold. Now let the \((n-r-1)\)-manifolds \( A_1, A_2, \ldots \) and the \((n-m+r)\)-manifolds \( B_1, B_2, \ldots \) be bases of the integral \((n-r-1)\)-cycles and the \((n-m+r)\)-cycles of \( N \). Let \( z_i, z'_i \) be the algebraic inverse of \( A_i \) and \( B_i \) respectively. Suppose that \( A_i \) and \( B_i \) are ordered in such a way that \( z_i \) is zero-divisor for \( i = 1, 2, \ldots, \alpha \) and only for these \( i \)'s, and that \( z'_i \) is zero-divisor for \( i = 1, 2, \ldots, \beta \) and only for these \( i \)'s. For all pairs \((i, j)\) satisfying \( i \leq \alpha \) and \( j \leq \beta \), now let \( \sigma_{ij} \) be Seifert's interlacing number of \((z_i, z'_j)\).

Then one again obtains a characteristic matrix \( (\sigma_{ij}) \) of \( f \) that possesses similar properties for the matrix of section 2.

References