97. An Observation on the Brown-McCoy Radical

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We wish to characterize in this note the Brown-McCoy radical $G(A)$ of an associative ring A , as a radical $(1, 1, 1, 1)(A)$, $(1, 1, 1, 0)(A)$, $(1, 1, 0, 1)(A)$ and $(1, 2, 1, 1)(A)$, respectively, where $(k, l, m, n)(A)$ is a well-defined special F -radical of the ring A in the sense of Brown-McCoy [3] for arbitrary nonnegative integers k, l, m and n . The concept of a (k, l, m, n) -radicalring A can be illustrated by the following elementary remarks. If the elements of A form on the operation $a \circ b = a + b - ab$ ($a, b \in A$) a Neumann-regular semigroup (for instance in the case of a Jacobson-radicalring A , when $(A, 0)$ is a group), then A is a $(k, 0, 1, 1)$ -radicalring and a $(0, l, 1, 1)$ -radicalring at the same time for any integers $k, l \geq 0$. Furthermore any (k, l, m, n) -semisimple ring A with minimum condition on *twosided* principal ideals is, as an (A, A) -doublemodule, completely reducible in a weak meaning, which generalizes the classical Wedderburn-Artin structure theorem also. (For the details of radicals, see [1], [2], [3].)

In this note the knowing of the results of Brown-McCoy [3] will be assumed for the reader. We denote the sum of all *twosided* principal ideals $(a^{(m)} \circ x \circ a^{(n)} - k \cdot a^{(l)})$ by $(k, l, m, n)(a)$, where a is a fixed element, x a varying element of A , $a \circ b = a + b - ab$, $a^{(0)} = 0$, $a^{(1)} = a$, $a^{(k+1)} = a^{(k)} \circ a$ and k, l, m, n are nonnegative integers. An element $a \in A$ is called (k, l, m, n) -regular, if $a \in (k, l, m, n)(a)$. We call an element $a \in A$ *strictly* (k, l, m, n) -regular, if any element b of the *twosided* principal ideal (a) generated by a is (k, l, m, n) -regular. The set $(k, l, m, n)(A)$ of all strictly (k, l, m, n) -regular-elements of A is called the (k, l, m, n) -radical of A . This is evidently a special F -radical of A [3]. The rings with (k, l, m, n) -radical (0) are called (k, l, m, n) -semisimple. We call a subdirectly irreducible (k, l, m, n) -semisimple ring A shortly: (k, l, m, n) -primitive. An element $a \neq 0$ with the condition $(k, l, m, n)(a) = 0$ is called here a (k, l, m, n) -distinguished element of A . By [3] the (k, l, m, n) -radical of A is the intersection of such ideals $\mathfrak{A}_\gamma$ ($\gamma \in \Gamma$) of A , that the factorrings $A/\mathfrak{A}_\gamma$ are (k, l, m, n) -primitive. $A/(k, l, m, n)(A)$ is (k, l, m, n) -semisimple, and a subdirect sum of (k, l, m, n) -primitive rings. By [3] a subdirectly irreducible ring A is (k, l, m, n) -primitive if and only if the minimal ideal $\mathfrak{D} \neq 0$ of A contains a (k, l, m, n) -distinguished element $d \neq 0$ playing the role of unity element in the case of radical

$(1, 1, 1, 1)(A) = G(A)$ of A .

Then holds the following

Theorem. An arbitrary (k, l, m, n) -primitive ring P has no proper twosided ideals, and we have $(1 - d^{(m)})P(1 - d^{(n)}) = 0$, $d = kd \cdot d^{(l)}$, $kd^{(l)} = d^{(m+n)}$ for a (k, l, m, n) -distinguished element $d (\neq 0)$ of P . Furthermore $G(A) = (1, 1, 1, 1)(A) = (1, 1, 1, 0)(A) = (1, 1, 0, 1)(A) = (1, 2, 1, 1)(A)$ are valid for the Brown-McCoy radical $G(A)$ of an arbitrary (associative) ring A .

Proof. If P is (k, l, m, n) -primitive, then there exists [3] a (k, l, m, n) -distinguished element $d \neq 0$ in the minimal ideal $\mathfrak{D} \neq 0$ of P . We have from $(k, l, m, n)(d) = 0$ evidently $d^{(m)} \circ x \circ d^{(n)} = k \cdot d^{(l)}$ for any $x \in P$. In the special case $X = 0$ follows $d^{(m+n)} = kd^{(l)}$ and thus in the case of arbitrary $x \in P$ is $X = d^{(m)} \cdot x + xd^{(n)} - d^{(m)}xd^{(n)} \in \mathfrak{D}$ valid. Therefore one has $P = \mathfrak{D}$ for the (k, l, m, n) -primitive rings P , and thus P cannot have proper twosided ideals. Obviously follows also $(1 - d^{(m)})P(1 - d^{(n)}) = 0$, $d = d \cdot d^{(m+n)}$ and $d = kd \cdot d^{(l)}$ respectively. Let A be now an arbitrary associative ring. Then $(1, 1, 1, 1)(A) = G(A)$ will be proved by showing, that any $(1, 1, 1, 1)$ -primitive ring P is a simple ring with unity element, and a similar fact holds for other special k, l, m, n mentioned in the above theorem. In the four cases k, l, m, n mentioned above, $k = 1$, hence $d = d \cdot d^{(l)}$ and $d^{(l)} = d^{(m+n)}$. If $l = m = n = 1$, then one has $d^2 = d$ for the (k, l, m, n) -distinguished element $d \neq 0$ of the (k, l, m, n) -primitive ring P . By $(1 - d)P(1 - d) = 0$ follows $C = (1 - d)P + P(1 - d)P = 0$, since P is by $d^2 = d \neq 0$ semi-simple in the sense of Jacobson, and the ideal C is nilpotent. Thus $(1 - d)P = 0$, $P = dP$ ($d^2 = d$) and similarly $P = Pd$ too. Therefore one has $(1, 1, 1, 1)(A) = G(A)$. If $k = l = m = 1$ and $n = 0$, immediately follows

$$(1, 1, 1, 0)(a) = \sum_{x \in A} (a \circ x \circ a^{(0)} - a) = \sum_{x \in A} (X - ax) = (1 - a)A + A(1 - a)A,$$

and thus $(1, 1, 1, 0)(A) = G(A)$ by the definition of the Brown-McCoy radical $G(A)$ of A [3]. The case $k = l = n = 1$ and $m = 0$ is totally similar to the previous case. If $k = m = n = 1$ and $l = 2$, then one has $d = d \cdot d^{(2)}$ and thus $d - 2d^2 + d^3 = 0$. Then by $d = 2d^2 - d^3 \neq 0$ is surely $P^2 \neq 0$, i.e. P is semisimple in the sense of Jacobson by the want of proper ideals. By $(1 - d)P(1 - d) = 0$ and $P^2 \neq 0$ follows $C = (1 - d)P + P(1 - d)P = 0$, since C is a nilpotent twosided ideal of P . This means $(1 - d)P = 0$ and $P = dP$. From $(d - d^2)P = (1 - d)dP = 0$ follows by $P^2 \neq 0$ evidently $d^2 = d$, for a Jacobson-semisimple ring we have no annihilator $\neq 0$. Therefore d is a left unity element of $P (= dP)$, and similarly one has $P = Pd$ also, which proves the theorem.

Remarks. 1) Any (k, l, m, n) -semisimple ring with minimum condition on twosided principal ideals is the discrete direct sum of (k, l, m, n) -primitive rings (see for these rings the above theorem), and conversely.

2) If the elements of A form with the operation $a \circ b = a + b - ab$ a Neumann-regular semigroup, then A is a $(k, 0, 1, 1)$ -radicalring and a $(0, l, 1, 1)$ -radicalring too.

3) It can be proved $A = (0, 0, 0, 0)(A) = (k, 0, 0, 1)(A) = (0, l, 0, 1)(A) = (k, 0, 1, 0)(A) = (0, l, 1, 0)(A) = (2, 1, 1, 0)(A) = (2, 1, 0, 1)(A) = (2, 1, 1, 1)(A)$.

For instance, if P is a $(2, 1, 1, 1)$ -primitive ring, then holds $d^{(2)} = 2d^{(1)}$ and $(1-d)P(1-d) = 0$, consequently $2d - d^2 = 2d$, $d^2 = 0$ and $0 \neq d = d - 2d^2 + d^3 = (1-d)d(1-d) \in (1-d)P(1-d) = 0$, which is a contradiction. Therefore $P = 0$ and $(2, 1, 1, 1)(A) = A$.

4) Any $(k, 0, 1, 1)$ -primitive ring P and any $(0, l, 1, 1)$ -primitive ring P are simple rings with unity element and with the condition $2P = P \neq 0$.

5) Any $(3, 1, 1, 1)$ -primitive ring, any $(3, 1, 1, 0)$ -primitive ring and any $(3, 1, 0, 1)$ -primitive ring P are simple rings with unity element and with the condition $2P = 0$. Therefore for example a $(3, 1, 1, 1)$ -primitive ring $P \neq 0$ cannot be for instance a $(0, l, 1, 1)$ -primitive ring.

6) We have seen $(1, 2, 1, 1)(A) = G(A)$. Then holds $(1, 2, 1, 1)(a) = ((1-a)A(1-a)) = (1-a)A(1-a) + A(1-a)A(1-a) + (1-a)A(1-a)A + A(1-a)A(1-a)A \supseteq W(a) = A(1-a)A(1-a)A$. The following W -regularity: $b \in W(b)$ determines a special F -radical $W(A)$ of A . If P is a W -primitive ring i.e. a W -semisimple and subdirectly irreducible ring, and if $P^3 \neq 0$, then P is a simple ring with unity element. If P is a W -primitive ring and if $P^2 = 0$, then the additive group P^+ is isomorphic to a group $C(p^k)$, where $1 \leq k \leq \infty$. If finally $P^2 \neq 0$ but $P^3 = 0$, and P is a W -primitive ring, then we have $P\mathfrak{D} = \mathfrak{D}P = 0$ for the minimal ideal $\mathfrak{D}$ of P and $(P^2)^+ \cong C(p^k)$ holds ($1 \leq k \leq \infty$). For example $A = \{a_1, a_2, \dots; b_1, b_2, \dots\}$ with $a_i^2 - b_i = pa_1 = b_i - pb_{i+1} = a_i a_j = a_i^3 = 0$ is a W -primitive ring with $A^3 = 0$ and $A^2 \neq 0$, $(A^2)^+ \cong C(p^\infty)$ ($i \neq j$).

7) Let A be an associative ring, M a right A -module and $\mathfrak{M}$ an arbitrary cardinal number. An A -submodule K of M is called $\mathfrak{M}$ -homoperfect, if the following conditions are satisfied:

I) $MA + K = M$;

II) M/K is a completely reducible A -module of dimension $< \mathfrak{M}$;

III) M/K has no proper A -submodule, which is invariant for all A -endomorphism of M/K ;

IV) if φ is an A -homomorphism of M/L onto M/K for an A -submodule L with the conditions I), II) and III), then φ is an isomorphism.

Let $\mathfrak{R}_m(M)$ be now itself M , if M has no proper $\mathfrak{M}$ -homoperfect submodules. If there exist in M proper $\mathfrak{M}$ -homoperfect submodules K_γ ($\gamma \in I'$), then we define $\mathfrak{R}_m(M) = \bigcap_{\gamma} K_\gamma$. In the case of $1 \in A$, a unitary

A -module M and $\mathfrak{M}=2$; $\mathfrak{R}_m(M)$ is the Bourbaki-radical of M [2], and in the case $\mathfrak{M}=2$ and arbitrary A we obtain the Kertész-radical of M [5]. We have proved solving in [6] a problem of Dr. A. Kertész [5] that the Jacobson-radical $\mathfrak{J}(A)$ of A must not coincide with the radical $\mathfrak{R}_2(A)$ of the right A -module A , if the power $|A|$ of A is no quadratfree finite cardinal number. We have generally only $\mathfrak{R}_2(A) \subseteq \mathfrak{J}(A)$. If in the ring A with left unity element holds the minimum condition on principal right ideals [7] and $\mathfrak{M}=\aleph_0$, then one has evidently $\mathfrak{R}_{\aleph_0}(A) \subseteq G(A)$ for the above radical $\mathfrak{R}_m(A)$ of the right A -module A and the Brown-McCoy radical $G(A)$ of A .*) Now we arise the following

Problem. What is a necessary and sufficient condition concerning A for the validity of $\mathfrak{R}_{\aleph_0}(A)=G(A)$? (Solve a similar problem of A. Kertész on $\mathfrak{R}_2(A)$ and $\mathfrak{J}(A)$ too!)

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*) It may be remarked that the theory of F -radicals can be formulated for A -modules too, where F is a well-defined mapping of any A -module M onto a set of submodules $F(m)$ of M ($m \in M$, $F(m) \subseteq M$) with the condition $F(m)\varphi = F(m\varphi)$ for any A -homomorphism φ of M . Then $m \in M$ is F -regular in the case $m \in F(m)$. Then the F -radical of M is the set $[m; m \in M, n \in F(n), n \in \{m\}]$.