A Remark on Convexity Theorems for Fourier Series

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In the previous paper [1], we have proved a number of convexity theorems concerning Fourier series. In the present paper, we shall improve some of them replacing either of the conditions by one-sided one.

Let \( \varphi(t) \) be an even function integrable in \((0, \pi)\) in Lebesgue sense, periodic of period \(2\pi\), and let

\[
\varphi(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nt,
\]

and

\[
\Phi_\alpha(t) = \varphi(t), \quad \Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \varphi(u) du \quad (\alpha > 0).
\]

The \((C, \beta)\) sum of the Fourier series of \( \varphi(t) \) at \( t = 0 \) is

\[
s_n = A_\beta \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_\beta a_n = \sum_{n=0}^{\infty} A_{\beta-1} s_n \quad (-\infty < \beta < \infty),
\]

where \( s_n = s^n_\beta \), \( A_\beta \) is defined as follows.

In what follows we understand that \( t \to 0 \) means \( t > 0 \) and \( t \to 0 \).

Now, Theorems 2, 4, 5, and 6 in the paper [1] can be improved as follows.

**Theorem 2'.** Let \( 0 < b \leq r - c \) and \( |c - b| < 1 \). If as \( t \to 0 \),

\[
(1) \quad \int_0^t |\Phi_\beta(u)| du = o(t^{r+1})
\]

and

\[
\int_0^t (|\Phi_\beta(u)| - \Phi_\beta(u)) du = O(t^{r+1}),
\]

then we have

\[
s_n = o(n^q), \quad q = b + (r-c) \frac{\beta - b}{(r-c) + b - \beta},
\]

as \( n \to \infty \), for \( c < r < r' \), where

\[
\gamma' = \inf \left( \gamma, \frac{(b+1)r - (\beta + 1)c}{r - c + b - \beta} \right).
\]

**Corollary 2.1'.** Let \( 0 < \beta < \gamma \) and \( 0 < \delta < 1 \). If \( (1) \) holds, and \( \varphi(t) = O_e(t^{r-1}) \), then

\[
s_n = o(n^\alpha), \quad \alpha = \beta \delta/(\gamma - \beta \delta).
\]

**Theorem 4'.** Let \( -1 \leq \beta, \ 0 \leq c \) and \( 0 < r + 1 - c \leq \beta + 1 - b \),
If as $n \to \infty$,

$$\sum_{r=0}^{n} |s^r_0| = o(n^{r+1})$$

and

$$\sum_{r=0}^{2n} (|s^r_{-1}| - s^r_{-1}) = O(n^r),$$

then we have

$$\Phi_r(t) = o(t^q), \quad q = b + (r - c) \frac{\beta + 1 - b}{\gamma + 1 - c},$$

as $t \to 0$, for $c < r < \gamma + 1$.

**Corollary 4.1'**. Let $0 < \delta < 1$ and $- (1 - \delta) < \gamma < \beta$. If (2) holds, and $a_n = O_L(n^{-(1-\delta)})$, then

$$\Phi_r(t) = o(t^q), \quad a = \delta(\beta + 1)/ (\beta - \gamma + \delta).$$

**Theorem 5'**. Let $0 \le b$ and $0 < \beta - b \le \gamma - c$, $[(b - 1)\gamma < \delta(\beta + 1)]$.

If

$$\Phi_\delta(t) = o(t^q) \quad \text{as} \quad t \to 0,$$

and

$$\sum_{v=n}^{2n} (|s^v_{-1}| - s^v_{-1}) = O(n^v) \quad \text{as} \quad n \to \infty,$$

then we have

$$\Phi_r(t) = o(t^q), \quad q = c + (r - b) \frac{\gamma - c}{\beta - b},$$

as $t \to 0$, for $b < r < \beta$.

**Corollary 5'**. Let $0 < \delta < 1$ and $\delta < \beta < \gamma$. If (3) holds, and $a_n = O_L(n^{-(1-\delta)})$, then

$$\Phi_r(t) = o(t^q), \quad a = \delta(\beta - c)/ (\beta - \gamma + \delta).$$

**Theorem 6'**. Let $0 \le c$, $0 < \gamma - c \le \beta - b$ and $|b - c| < 1$, $[c(\beta + 1) < (b + 1)\gamma]$. If

$$s^\alpha_n = o(n^r) \quad \text{as} \quad n \to \infty,$$

and

$$\int_0^t (|\Phi_c(u)| - \Phi_c(u))du = O(t^{\nu + 1}) \quad \text{as} \quad t \to 0,$$

then we have

$$s^\alpha_n = o(n^r), \quad q = c + (r - b) \frac{\gamma - c}{\beta - b},$$

as $n \to \infty$, for $b < r < \beta$.

**Corollary 6'**. Let $0 < \gamma < \beta$ and $0 < \delta < 1$. If (4) holds, and $\varphi(t) = O_L(t^{-1})$, then

$$s^\alpha_n = o(n^r), \quad \alpha = \delta(\beta - \gamma + \delta).$$

**Proof of Theorem 6'**. Using the number $\gamma'$ such as $\gamma' - c = \beta - b$ the assumptions imply

$$\gamma' > 0 \quad \text{and} \quad s^\alpha_n = o(n^r).$$

So, by a theorem of Izumi [2], i.e. Corollary 4.2 in [1], we have

$$\Phi_\gamma(t) = o(t^{\gamma + 1 + \delta}), \quad \varepsilon > 0.$$ Consequently
(6) \( \Phi_{c+1}(t) = o(t^{k+b}) \)
holds for every \( k > b - 1 \). On the other hand, the condition (5) implies for \( 0 < t \leq t_0 \)
\[ \int_0^{t_0} (|\Phi_c(u)| - \Phi_c(u)) du \leq At^{b+1}, \]
\( A \) being an absolute constant, and then
(7) \( \Phi_{c+1}(t+u) - \Phi_{c+1}(t) \geq -At^{b+1}, 0 < u < t. \)
From (6) and (7) we have \( \Phi_{c+1}(t) = O(t^{b+1}) \) by Theorem 8 in [1], and
so (5) yields
(8) \[ \int_0^t |\Phi_c(u)| du = O(t^{b+1}). \]
The result follows from (4) and (8). Cf. Theorem 6 and Lemmas 3 and 3' in [1].
The proofs of Theorems 2', 4', and 5' are similar.

References