5. **On a Boundary Theorem on Open Riemann Surfaces**

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1. **Introduction.** Let $U$ be the class of Riemann surfaces on which there exist the Green function and at least a bounded minimal positive harmonic function (C. Constantinescu and A. Cornea [1]) and $O_l$ be the class of Riemann surfaces on which there exist the Green function and no non-constant Lindelöfian meromorphic function (M. Heins [3]). Let $R$ be an open Riemann surface and $Q$ be a subregion of the Riemann surface $R$ whose relative boundary $\partial Q$ with respect to $R$ consists of at most an enumerable number of analytic curves clustering nowhere in $R$. If there exists no non-constant single-valued bounded harmonic function in $Q$ which vanishes continuously on $\partial Q$, we say that $Q$ belongs to $SO_{HB}$. The following theorem was proved by many authors (see [2], [4], [6], and [8]).

Let $R$ be an open Riemann surface belonging to the class $U$ and $Q$ be a subregion of $R$ which satisfies the above boundary condition and does not belong to $SO_{HB}$, then $Q$ belongs to $O_l$.

In the present paper we shall give another simple proof of this assertion with aid of the notion of thinness in Martin’s space [5] (which is given by Martin’s compactification of an open Riemann surface), introduced by L. Naïm [7].

2. **Preliminaries.** We shall introduce the notion of thinness and some useful results for our purpose.

Let $R$ be an open Riemann surface and $\hat{R}$ be Martin’s space associated with $R$. We say that $\Delta^a=\hat{R}-R$ is the Martin boundary of $R$. Now let $K_x(y)$ be a kernel function in the sense of Martin, that is $K_x(y)=\frac{G(x, y)}{G(x, y_0)}$ for $x\in\hat{R}-\{y_0\}$, $y\in R$ with a fixed point $y_0$ in $R$. Then $x_0$ is said to be a minimal point of $\Delta^a$ if $K_{x_0}(y)$ is a minimal positive harmonic function in $R$ in the sense of Martin and $x_0$ is said to be a bounded minimal point of $\Delta^a$ if, in addition, $K_{x_0}(y)$ is bounded in $R$.

Let $m$ be a positive measure in $R$, then a $K$-potential with respect to the measure $m$ in $R$ is defined in $\hat{R}-\{y_0\}$ by

$$U(x)=\int K_x(y) \, dm(y).$$

Definition. A subset $E$ of $R$ is said to be thin at a point $x_0$ in
$\hat{\mathcal{R}}-\{y_0\}$ if $x_0$ is not a limit point of $E$ or otherwise if there is a $K$-potential such that

$$U(x_0) < \liminf_{x \to x_0, x \in \hat{\mathcal{R}}} U(x).$$

Then we can immediately see that the union of a finite number of the thin sets at $x_0$ is also thin.

Naïm [7] proved the following:

(2.1) $\mathcal{R}$ is not thin at any minimal point $x_0$ of $\mathcal{A}^\mathcal{R}$ and vice versa (Theorem 3).

(2.2) A set $E$ of $\mathcal{R}$ is thin at a minimal point $x_0$, if and only if the extremization $\mathcal{C}_{K_{x_0}}^E$ of the kernel function $K_{x_0}(y)$ over $\mathcal{R}-E$ does not conserve this function, that is,

$$\mathcal{C}_{K_{x_0}}^E(y) \equiv K_{x_0}(y)$$

(Theorem 5).

Here the notion of the extremization is the following:

The extremization $\mathcal{C}_v^E$ of the positive superharmonic function $v$ over the set $E$ is the least positive superharmonic function which dominates $v$ in $\mathcal{R}-E$ except for a set of capacity zero.

(2.3) Let $u$ be a harmonic function in $\mathcal{R}$, $\Omega$ be an open set of $\mathcal{R}$ and $\Omega^*$ be a boundary of $\Omega$ with respect to Martin’s space $\mathcal{R}$. Let $u$ be the function on $\Omega^*$ which coincides with $u$ on $\Omega \cap \mathcal{R}$ and 0 on $\Omega \cap \mathcal{A}^\mathcal{R}$ and $H_\Omega^0(y)$ be the solution of Dirichlet problem with respect to $\Omega$ in the sense of Brelot.

Let $x_0$ be a point of $\Omega$ being minimal in $\mathcal{A}^\mathcal{R}$. If $u=K_{x_0}(y)$ is different from $H_\Omega^0(y)$, then the difference $u(y)-H_\Omega^0(y)$ is a minimal positive harmonic function in $\Omega$ (Theorem 12).

On the other hand, Heins [3] proved the following assertion:

Let $f$ be a single-valued meromorphic function in $\mathcal{R}$, and $\mathcal{C}$ be a subset of the $w$-sphere. For each open set $\delta$ of the $w$-sphere, we shall denote the greatest harmonic minorant of the extremization of the constant 1 over $\mathcal{R}-f^{-1}(\delta)$ by $\mathcal{E}_{1}^{\mathcal{R}-f^{-1}(\delta)}(y)$ and the lower envelope of the family $\{\mathcal{E}_{1}^{\mathcal{R}-f^{-1}(\delta)}(y)\}_{\delta} \subset B_\delta$.

(2.4) If $f$ is Lindelöfian, then $\text{Cap } \mathcal{C}=0$ implies $B_\delta=0$.

3. Theorems. Using these results we shall prove the following theorem:

Theorem 1. Let $\mathcal{R}$ be an open Riemann surface belonging to the class $U$, then $\mathcal{R}$ belongs to the class $O_L$.

Proof. Suppose that there exists an open Riemann surface $\mathcal{R}$ which belongs to the class $U$ and does not belong to the class $O_L$. Let $f$ be a non constant Lindelöfian meromorphic function in $\mathcal{R}$ and $x_0$ be a bounded minimal point of the Martin boundary $\mathcal{A}^{\mathcal{R}}$.

On the other hand we can consider as $\mathcal{C}$ a single point $w$ of the
w-sphere and as \( \delta \) an open neighborhood \( V(w) \) of the \( w \), so \( B_\delta \) coincides with the lower envelope of the family \( \{ \mathcal{E}_{R-f^{-1}(V(w))}(y) \} \).

Now we see that
\[
\mathcal{E}_{R-f^{-1}(V(w))}(y) \geq k \cdot \mathcal{E}_{K_{x_0}}(y),
\]
where \( k = 1/\sup_{R} K_{x_0}(y) > 0 \), since \( K_{x_0}(y) \) is bounded in \( R \).

Then there exists a small neighborhood \( V(w) \) of \( w \) such that
\[
\mathcal{E}_{R-f^{-1}(V(w))}(y) \equiv K_{x_0}(y),
\]
therefore \( f^{-1}(V(w)) \) is thin at \( x_0 \) by (2.2).

In fact if we assume that for any \( V(w) \)
\[
\mathcal{E}_{R-f^{-1}(V(w))}(y) \equiv K_{x_0}(y)
\]
by the definition of the greatest harmonic minorant, we have
\[
\mathcal{E}_{R-f^{-1}(V(w))}(y) \geq k \cdot \mathcal{E}_{K_{x_0}}(y) \equiv K_{x_0}(y)
\]
and
\[
\mathcal{E}_{R-f^{-1}(V(w))}(y) \geq k \cdot K_{x_0}(y) = k > 0 \quad \text{for any } V(w).
\]

For a small positive number \( \varepsilon \) \((\varepsilon < k)\) there exists a small \( V(w) \) such that
\[
\mathcal{E}_{R-f^{-1}(V(w))}(y_0) < \varepsilon,
\]
since \( B_{1/\varepsilon} = 0 \) by (2.4). This is impossible.

Thus for any point \( w \) of the \( w \)-sphere we can choose an open neighborhood \( V(w) \) of \( w \) such that \( f^{-1}(V(w)) \) is thin at \( x_0 \).

The family \( \{ V(w) \}_{w \in w \text{-sphere}} \) is an open covering of the \( w \)-sphere and we can choose a finite number of \( V(w_i) \) \((i = 1, \ldots, n)\) such that \( \{ V(w_i) \}_{i=1}^n \) is a covering of the \( w \)-sphere by the compactness of this.

Every \( f^{-1}(V(w_i)) \) is thin at \( x_0 \), so \( \bigcup_{i=1}^n f^{-1}(V(w_i)) \) is also thin at the point \( x_0 \) of \( \Delta^R \). But this set coincides with \( R \). This contradicts (2.1) and leads to our assertion.

As a consequence of Theorem 1 we have

**Theorem 2.** Let \( R \) be an open Riemann surface belonging to the class \( U \) and \( \Omega \) be a subregion of \( R \) such that \( R-\Omega \) is thin at some bounded minimal point \( x_0 \) of the Martin boundary \( \Delta^R \), then \( \Omega \) belongs to the class \( O_L \).

**Proof.** We know that \( H_{K_{x_0}}^g(y) \equiv \mathcal{E}_{K_{x_0}}(y) \) in \( \Omega \). Since \( R-\Omega \) is thin at \( x_0 \), \( \mathcal{E}_{K_{x_0}}(y) \equiv K_{x_0}(y) \).

Then by the property of the extremization we see that \( K_{x_0}(y) - H_{K_{x_0}}^g(y) > 0 \) in \( \Omega \), and by (2.3) \( K_{x_0}(y) - H_{K_{x_0}}^g(y) > 0 \) is a bounded minimal harmonic function in \( \Omega \). This shows us that \( \Omega \) belongs to the class \( U \). We conclude by Theorem 1 that \( \Omega \) belongs to the class \( O_L \).

**References**


