26. Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. VI

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On the assumption that $C$ and $Q$ denote respectively a given complex number and an appropriately large circle with center at the origin and that the ordinary part $R(\lambda)$ of the function $S(\lambda)$ defined in the statement of Theorem 1 [1] is a transcendental integral function, in this paper we shall discuss the relation between the distribution of $\zeta$-points of $S(\lambda)$ and that of $\zeta$-points of $R(\lambda)$ in the exterior of the same circle $Q$ and shall then show that, if each of $S(\lambda)$ and $R(\lambda)$ has its finite exceptional value for the exterior of $Q$, the two exceptional values are identical under some conditions.

Theorem 16. Let $S(\lambda), R(\lambda),$ and $\{\zeta_n\}$ be the same notations as those in Theorem 1; let $\sigma$ be an appropriately large number such that $\sup |\lambda_n| < \sigma < \infty$; let $\{\zeta_n\}$ be an infinite sequence of all $\zeta$-points of $R(\lambda)$ in the exterior of the circle $|\lambda| = \sigma$ such that

$$R(z_n) = \zeta \quad (n = 1, 2, 3, \ldots)$$

and $|z_n| \to \infty$ ($n \to \infty$), each $\zeta$-point being counted with the proper multiplicity; let

$$C = \sup \left\{ \frac{1}{2\pi} \left| \int_0^{2\pi} S(\rho e^{i\theta}) e^{i\nu n \theta} d\theta \right| \right\} (< \infty),$$

where $\rho$ is an arbitrarily prescribed number subject to the condition $\sup |\lambda_n| < \rho < \infty$; let $\mu$ be the greatest value of the positive integers $\nu_n$ in the first non-zero coefficients $R(z_n)/\nu_n!$ of the Taylor expansions of $R(\lambda)$ at $z_n$, $n = 1, 2, 3, \ldots$; let $m = \inf \{|R(z_n)/\nu_n!|\}$ be positive; let $M = \sup \left\{ \max \{|R^{(k)}(z_n)|/k!|\} \right\}$ ($n, k = 1, 2, 3, \ldots$) be finite; and let $r$ be an arbitrarily given number such that $0 < r < m/(M + m)$.

Then, in the interior of the circle $|\lambda - z_n| = r$ associated with any $z_n$ satisfying

$$\left\{ \frac{C}{r^n \left( m - \frac{Mr}{1 - r} \right)} + 1 \right\}^\rho + r < |z_n|,$$

$S(\lambda)$ has $\zeta$-points whose number (counted according to multiplicity) equals that of $\zeta$-points of $R(\lambda)$ in the interior of the same circle as it.

Proof. It must first be noted that the case where $R(\lambda)$ has such $\zeta$-points $\{z_n\}$ as was described in the statement of the present theorem
can occur in accordance with Picard's theorem when it is a trans-
cendental integral function.

Now, by hypotheses,

\[ |R(z_n + re^{i\theta}) - \zeta| = \left| \sum_{k=1}^{\infty} \frac{R^{(k)}(z_n)}{k!} (re^{i\theta})^k \right| \geq r^{n_1} \left( m - \frac{Mr}{1-r} \right) \geq r^n \left( m - \frac{Mr}{1-r} \right) > 0, \]

where \( n_1 \) is the same notation as that defined in the statement of
the present theorem; and in addition, denoting by \( \chi(\lambda) \) the sum of
the two principal parts of \( S(\lambda) \) and applying the expansions of \( R(\lambda) \)
and \( S(\lambda) \)[2], we can find at once that for every \( z_n \) satisfying \( |z_n| > r + p \)

\[ |\chi(z_n + re^{i\theta})| = \frac{1}{2} \left| \sum_{k=1}^{\infty} (a_k + ib_k) \left( \frac{\rho}{z_n + re^{i\theta}} \right)^k \right| \leq \frac{1}{2} \sum_{k=1}^{\infty} |a_k + ib_k| \left| \frac{\rho}{z_n - r} \right|^k \leq \frac{C}{|z_n| - r - \rho} < \infty, \]

where

\[ a_k = \frac{1}{\pi} \int_0^{2\pi} S(\rho e^{i\theta}) \cos kt \, dt \]
\[ b_k = \frac{1}{\pi} \int_0^{2\pi} S(\rho e^{i\theta}) \sin kt \, dt. \]

Since, on the other hand, there exist large positive integers \( n \)
such that

\[ 0 < \frac{C}{|z_n| - r - \rho} < r^p \left( m - \frac{Mr}{1-r} \right), \]

i.e., \( \left\{ \frac{C}{r^p \left( m - \frac{Mr}{1-r} \right)} + 1 \right\}^p + r < |z_n|, \]

by denoting by \( G \) the least value of \( n \) satisfying this last inequality we
obtain the inequalities \( |R(z_{n+p} + re^{i\theta}) - \zeta| > |\chi(z_{n+p} + re^{i\theta})|, p = 0, 1, 2, \ldots, \)
for every \( \theta \) in the closed interval \([0, 2\pi]\). If, for simplicity, we denote
by \( \Gamma_p \) the circle \( |\lambda - z_{n+p}| = r \) associated with the point \( z_{n+p} \) for each
value of \( p = 0, 1, 2, \ldots, \) then the just established result shows that \( |R(\lambda) - \zeta| > |\chi(\lambda)| \) on \( \Gamma_p, p = 0, 1, 2, \ldots, \) In addition to it, \( R(\lambda) - \zeta \) and \( \chi(\lambda) \)
are both regular inside and on any \( \Gamma_p \) by the condition \( |z_{n+p}| > r + p. \)
In consequence, it is found with the help of Rouche's theorem that the function
\( S(\lambda) - \zeta = [R(\lambda) - \zeta] + \chi(\lambda) \) has zeros (with multiplicities properly counted) inside any \( \Gamma_p \) and that the number of those zeros is
equal to that of zeros (with multiplicities properly counted) of \( R(\lambda) - \zeta \)
inside the same \( \Gamma_p. \) Evidently this implies that the result stated
in the present theorem holds true.

Theorem 17. Let \( S(\lambda), R(\lambda), \{\zeta_\alpha\}, \sigma, \rho, C, \mu, M, \) and \( r \) be the same notations as those in Theorem 16 but let \( \{z_n\} \) in it be an infinite sequence of all \( \zeta \)-points of \( S(\lambda) \) in the exterior of the circle \( |\lambda|=\sigma \) such that

\[
S(\lambda) = \zeta \\
\sigma < |z_n| \leq |z_{n+1}| \quad (n=1, 2, 3, \ldots)
\]

and \( |z_n| \to \infty \) \( (n \to \infty) \), each \( \zeta \)-point being counted with the proper multiplicity; and let \( \varepsilon \) be a positive number less than \( r^\sigma \left( \frac{M \rho}{1-r} \right) \).

Then, in the interior of the circle \( |\lambda - z_n| = r \) associated with any \( z_n \) satisfying the conditions \( |R(z_n) - \zeta| < \varepsilon \) and

\[
\left\{ \frac{2C}{r^\sigma \left( \frac{M \rho}{1-r} \right) - \varepsilon} + 1 \right\} \rho + r < |z_n|,
\]

\( R(\lambda) \) has \( \zeta \)-points whose number (counted according to multiplicity) equals that of \( \zeta \)-points of \( S(\lambda) \) in the interior of the same circle as it.

Proof. As will be seen immediately from the expansion of \( \chi(\lambda) \) \( [2] \), \( |\chi(\lambda)| \to 0 \) \( (|\lambda| \to \infty) \) and so \( |R(z_n) - \zeta| \to 0 \) \( (n \to \infty) \) by virtue of the hypothesis \( S(z_n) = \zeta, \) \( \omega=1, 2, 3 \ldots \). Since, moreover, by hypotheses,

\[
|R(z_n + r e^{i\theta} - \zeta| \geq r^\sigma \left( \frac{M \rho}{1-r} \right) - |R(z_n) - \zeta|\\
> r^\sigma \left( \frac{M \rho}{1-r} \right) - \varepsilon
\]

for all \( z_n \) with \( |R(z_n) - \zeta| < \varepsilon \), and since, as demonstrated in the course of the proof of Theorem 16,

\[
|\chi(z_n + r e^{i\theta})| \leq \frac{C \rho}{|z_n| - r - \rho} < \infty
\]

for any \( z_n \) with \( |z_n| > r + \rho \), it can be verified without difficulty from the relation \( S(z_n + r e^{i\theta} - \zeta) = [R(z_n + r e^{i\theta} - \zeta) + \chi(z_n + r e^{i\theta})] \) that \( |S(z_n + r e^{i\theta}) - \zeta| > |\chi(z_n + r e^{i\theta})| \) for every \( \theta \in [0, 2\pi] \) and every \( z_n \) satisfying the conditions \( |R(z_n) - \zeta| < \varepsilon \) and

\[
0 < \frac{2C \rho}{|z_n| - r - \rho} < r^\sigma \left( \frac{M \rho}{1-r} \right) - \varepsilon, \text{ i.e.,}
\]

\[
\left\{ \frac{2C}{r^\sigma \left( \frac{M \rho}{1-r} \right) - \varepsilon} + 1 \right\} \rho + r < |z_n|.
\]

For any \( z_n \) satisfying these two conditions, we have therefore the inequality \( |S(\lambda) - \zeta| > |\chi(\lambda)| \) holding on the circle \( |\lambda - z_n| = r \), and moreover \( S(\lambda) - \zeta \) and \( \chi(\lambda) \) are both regular inside and on this circle by the condition \( |z_n| > r + \rho \). On the other hand, as can be seen from
the familiar method of the proof of the Rouché theorem quoted before, it is rewritten as follows: if \( f(\lambda) \) and \( g(\lambda) \) are both regular on a simply connected domain \( D \), if \( \Gamma \) is the curve defined by the equation \( \lambda = \xi(s), 0 \leq s \leq 1, \xi(0) = 0, \xi(1) = 1 \), where \( \xi(s) \) is a continuous function of \( s \), and if for any point \( \xi \) on \( \Gamma \) the function \( f(\lambda) - \xi(s)g(\lambda) \) never vanishes on a rectifiable closed Jordan curve \( K \) contained in \( D \), then, in the interior of \( K \), the number (counted according to multiplicity) of zeros of \( f(\lambda) - g(\lambda) \) coincides with that of zeros of \( f(\lambda) \).

In consequence, by applying this rewritten Rouché theorem to the above established results, we can conclude that the number (counted according to multiplicity) of zeros of \( f(\lambda) - g(\lambda) \) coincides with that of zeros of \( f(\lambda) \). In consequence, by applying this rewritten Rouché theorem to the above established results, we can conclude that the number (counted according to multiplicity) of zeros of \( f(\lambda) - g(\lambda) \) coincides with that of zeros of \( f(\lambda) \).

Theorem 18. Let \( S(\lambda), R(\lambda), \{\lambda_k\}, \) and \( \sigma \) have the same meanings as in Theorems 16 and 17 respectively. If \( S(\lambda) \) has \( \zeta(= \infty) \) as its exceptional value for the exterior of the circle \( |\lambda| = \sigma \), that is, if the equation \( S(\lambda) = \zeta \) has not infinitely many solutions in the domain \( \mathcal{D}[\lambda: |\lambda| > \sigma] \), then the same is also valid of the equation \( R(\lambda) = \zeta \), and conversely.

Proof. First we consider the case where \( S(\lambda) \) has \( \zeta \) as its finite exceptional value for the above-mentioned domain \( \mathcal{D} \). If, contrary to what we wish to prove, \( \zeta \) is not the exceptional value of \( R(\lambda) \) for \( \mathcal{D} \), there would exist \( \zeta \)-points \( \{z_n\}_n \) of \( R(\lambda) \), which are so arranged as to satisfy the conditions stated in Theorem 16. Contrary to the hypothesis on \( S(\lambda) \), this result would lead us to the conclusion that \( S(\lambda) \) has also an infinite sequence of \( \zeta \)-points in \( \mathcal{D} \), according to Theorem 16. Consequently \( \zeta \) must be the exceptional value of \( R(\lambda) \).

Next we consider the case where \( \zeta \) is the finite exceptional value of \( R(\lambda) \). In this case, by making use of a method analogous to that applied in the preceding paragraph and of Theorem 17 it can be verified similarly that \( S(\lambda) \) has \( \zeta \) as its exceptional value for the domain \( \mathcal{D} \).

The proof of the theorem is thus complete.

Remark. We here remark on \( R^{(k)}(\lambda), k = 0, 1, 2, \ldots \), that each of these functions is expressible by a curvilinear integral associated with \( S(\lambda) \) itself, as shown in Theorem 1.
References


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For \[ \frac{1}{(1-\mu)^{p^*}} M_\delta(\rho, 0) = K \] read \[ \frac{1}{(1-\mu)^{p^*}} M_\delta(\rho, 0) = K \].