19. The $\varepsilon$-Entropy of Some Classes of Harmonic Functions

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1. Let $K$ be a bounded continuum in $q$-dimensional Euclidian space and $G$ be a bounded open set containing $K$. For complex-valued function $u(x)$ in $G$, we define $|u(x)| = \sup_{x \in K} |u(x)|$. We consider classes $H_\varepsilon(C)$ of functions $u(x)$ which are harmonic in $G$ and bounded in $G$ by the constant $C$. When we introduce the metric $|| \cdot ||$ in $H_\varepsilon(C)$, we shall denote it by $H_\varepsilon^C(C)$.

The purpose of the present paper is to compute "$\varepsilon$-entropy" and "$\varepsilon$-capacity" of $H_\varepsilon^C(C)$ for some $K$ and $G$. The exact formulae for them are given in 3. Using these results, we can compute the "functional dimension" of the vector space of harmonic function in 4.

The problem of computing $\varepsilon$-entropy of the space of solutions of partial differential equations was posed by Prof. H. Yoshizawa.

2. Following [3], we shall list definitions which are necessary to state our results. Let $R$ be a metric space and $A$ a set in $R$.

**Definition 1.** A set $B$ in $R$ is called an $\varepsilon$-net for the set $A$ if every points of $A$ is at a distance not exceeding $\varepsilon$ from some point of $B$.

**Definition 2.** A set $B$ in $R$ is called $\varepsilon$-separated if the distance of any distinct points of $B$ are greater than $\varepsilon$.

Now we assume the set $A$ is totally bounded.

**Definition 3.** $N(\varepsilon, A)$ is the minimal number of points in all possible $\varepsilon$-net for $A$. $H(\varepsilon, A) = \log N(\varepsilon, A)$ is called $\varepsilon$-entropy of the set $A$. ($\log N$ will always denote the logarithm of the number $N$ in the base 2.)

**Definition 4.** $M(\varepsilon, A)$ is the maximal number of points in all possible $\varepsilon$-separated subsets of the set $A$. $C(\varepsilon, A) = \log M(\varepsilon, A)$ is called the $\varepsilon$-capacity of $A$.

We shall state a simple theorem which will be used later [3].

**Theorem 1.** $M(2\varepsilon, A) \leq N(\varepsilon, A)$

3. Our result is as follows.

**Theorem.** Let $K_r = \{x; \sum_{i=1}^q x_i^2 < r^2\}$ and $G_r = \{x; \sum_{i=1}^q x_i^2 < R^2\}$ in $q$-dimensional space. Then

$H(\varepsilon, H_\varepsilon^{K_r}(C)) = \frac{4}{q!} (\log R/\varepsilon)^{q-1} \left( \log 1/\varepsilon \right)^q + O((\log 1/\varepsilon)^{q-1} \log \log 1/\varepsilon)$,

$C(2\varepsilon, H_\varepsilon^{K_r}(C)) = \frac{4}{q!} (\log R/\varepsilon)^{q-1} \left( \log 1/\varepsilon \right)^q + O((\log 1/\varepsilon)^{q-1} \log \log 1/\varepsilon)$.

(For notations, see 1 and 2.)

**Remark.** From Theorem 1 it is sufficient to estimate $H(\varepsilon, A)$ from
above (formula in 8) \( C(2s, A) \) from below (formula in 10).

4. Let \( G \) be an arbitrary domain in \( q \)-dimensional space. \( H_\theta \) is the totality of harmonic function in \( G \). We introduce in \( H_\theta \) compact uniform topology and consider it as linear topological space. The functional dimension of a linear topological space \( \Phi \) is defined as follows: (\([2]\))

\[
\text{df } \Phi = \sup_{V} \inf_{\varphi} \lim_{r \to 0} \log \log N(\epsilon, V/\varphi)/\log \log 1/\epsilon
\]

where \( N(\epsilon, V/\varphi) = \inf \{ N; V \subseteq \bigcup_{k=1}^{N} (\varphi_k + U), \varphi_k \in \Phi \} \) and \( \inf \) and \( \sup \) are taken for all neighbourhoods of 0 in \( \Phi \).

Then we obtain \( \text{df } H_\theta = q \).

In order to compute \( \text{df } H_\theta \) we use our results and the following properties for \( N(\epsilon, H_\theta^k(C)) \) which can be proved easily:

\[
N(\epsilon, H_\theta^k(C)) \leq N(\epsilon, H_\theta^k(C)) \quad \text{if} \quad K_1 \subset K_2 \quad \text{and} \quad G_1 \supseteq G_2
\]

\[
N(\epsilon, H_\theta^{k_1 + k_2}(C)) \leq N(\epsilon, H_\theta^k(C)) N(\epsilon, H_\theta^k(C)).
\]

5. We shall prove our THEOREM in 5-10. First we shall consider hyperspherical harmonics for the later use. Function \( u(x) = u(\rho, s) \) of the class \( A = H_\theta^k(C) \) can be expanded in hyperspherical harmonics in \( K_r \) (\([1]\)).

\[
u(\rho, s) = \sum_{l=0}^{\infty} (2l + p) (\rho/r)^l u_l(s)
\]

(1)

where \( q = p + 2 \), \( S(r) \) is the sphere of radius \( r \), \( \gamma = \angle sOs' \) and \((1 - 2ax + a^2)^{-p/2} = \sum_{r=0}^{\infty} a^r V^l(\gamma(x)) \).

We list here some properties of the above expansion for later use

(A) We have \( |V_l^{(p)}(\cos \gamma)| \leq c_l \), where \( c_l = V^{(p)}(1) = (l, p)/(1, p) \), \((\lambda, k) = \Gamma(\lambda + k)/\Gamma(\lambda) = \lambda(\lambda + 1) \cdots (\lambda + k - 1) \).

(B) Hyperspherical functions of order \( l \) form a \( d_l \)-dimensional vector space \( H_l \), where \( d_l = (l + 1, p - 1)/(1, p) \cdot (2l + p) \).

(C) We have \( \int V_l^{(p)}(\cos \angle sOs') ds = (4\pi^{p/2 + 1}/\Gamma(p/2) \cdot c_l/2l + p) \).

LEMMA. If \( y_i(s) \in H_l \) and \( \int |y_i(s)|^2 ds = 1 \), then

\[
|y_i(s)| \leq C_i ((2l + p) c_l)^{1/2}
\]

and \( C_i \) does not depend on \( l \) (\( C_i \) will always mean constants which depend only on \( p, r, R, C \)).

PROOF. Put \( u(\rho, s) = \rho \cdot y_i(s) \) in (1). If we use Schwartz' inequality and (C), we get (2).

6. We define a norm for bounded functions on unit sphere by \( ||u(s)||^2 = \sup_{s \in S(1)} |u(s)| \). Then we get two inequalities for expansion (1).

\[
||u_i(s)||^2 \leq C_{i}' ||u(\rho, s)||, \quad \text{where} \quad c_i' = [c_i/(2l + p)]^{1/2}
\]

(3)

\[
||u(\rho, s)|| \leq \sum_{l=0}^{\infty} (2l + p) ||u_i(s)||^2.
\]

(4)
Because \( u(p, s) \in A \) is harmonic in \( G_R \), it has an expansion of the form (1) in \( K_p(R'<R) \), where \( r \) is substituted by \( R' \). By equating \( \rho'^{s} \)'s coefficients in this expansion and in the original one, we get

\[
(5) \quad u_i(s) = \left( \frac{I(p/2)}{4\pi^{p/2} + 1} \right) \cdot R^{p-1} (r/R')^{i} \int_{\partial K_p(R')} u(\rho, s') V_i^{(p)} (\cos r) \, ds'.
\]

We obtain from (5) and \( |u(\rho, s)| \leq C \) in \( G_R \), \( ||u_i(s)||' \leq C_6 c_i (r/R')^{i} \). Because \( R'<R \) is arbitrary, we get finally

\[
(6) \quad ||u_i(s)||' \leq C_6 c_i e^{-\frac{h}{r}}, \quad \text{where } e^h = R/r.
\]

7. We define \( n \) as the smallest number that satisfies \( \sum_{l=0}^{\infty} (2l+p) \times C_6 c_i e^{-\frac{l}{r}} \leq \varepsilon/2 \). Because left side of the above inequality is smaller than \( C_4 n^N e^{-hN} \) for some \( N \), we get the following estimation of \( n \):

\[
(7) \quad n = \log \frac{1}{\varepsilon} / h \log e + O(\log \log 1/s).
\]

For such \( n \), if we define \( \hat{u}(\rho, s) \) by \( \hat{u}(\rho, s) = \sum_{l=0}^{n-1} (2l+p)(\rho/r)^{i} u_i(s) \) then \( \hat{A} = \{\hat{u} ; u \in A \} \) is an \( \varepsilon/2 \)-net for \( A \). We define \( A_i \) by \( \{u_i(s) ; u(\rho, s) \in A \} \). If we put \( \varepsilon' = \varepsilon/2 / n(n+p-1) \) and if we construct \( \varepsilon' \)-net \( B_i \) for \( A_i \) in \( H \) (in metric \( ||\cdot||' \) ), then \( \sum_{l=0}^{n-1} (2l+p)(\rho/r)^{i} u_i(s) ; u_i \in B_i \) will be \( \varepsilon/2 \)-net for \( \hat{A} \), so \( \varepsilon \)-net for \( A \).

If number of elements \( B_i \) is \( N_i \), \( N(\varepsilon, A) \leq \prod_{i=0}^{\infty} N_i \).

8. We construct \( B_i \) and evaluate \( N_i \). Let \( \{y_k(s), 1 \leq k \leq d_1 \} \) be complete orthonormal system in \( H \), and we shall expand \( u_i(s) \in A_i \), in \( \{y_k(s)\} \):

\[
u_i(s) = \sum_{k=1}^{d_1} b_k y_k(s), \quad \text{where } b_k = \int_{S} u_i(s) y_k^2(s) \, ds.
\]

From (6), we obtain for \( u_i(s) \in A_i \)

\[
(8) \quad |b_k| \leq C_6 c_i e^{-\frac{k}{r}}.
\]

If we consider the class of elements of \( H \), whose \( b_k \) can be written as \( b_k = m_k \delta + m'_k \delta / 1 \) (where \( m_k, m'_k \) are integers, and \( \delta = (2\varepsilon' / \sqrt{2}) / d_1 C_6 c_i (2l+p)c_i^{1/k} \)), then from the lemma, it is an \( \varepsilon \)-net for \( A \).

From (8), it is sufficient to choose \( |m_k| \leq C_4 c_i e^{-\frac{k}{r}} / \delta \). So

\[
N_i \leq \left[ \frac{m_k c_i e^{-\frac{k}{r}}}{\delta} \right]^{2d_1}.
\]

\[
H(\varepsilon, A) \leq \sum_{i=0}^{\infty} \log N_i = \sum_{i=0}^{n-1} 2d_i \log (C_5 n(n+p-1) d_i c_i e^{-h/s}).
\]

\[
= \frac{4}{(p+2)!} \log (1/s)^{p+2} / (h \log e)^{p+1} O((\log (1/s))^{p+1} \log \log 1/s).
\]

9. We now derive lower estimate for \( C(2s, A) \). For this purpose we use two facts:

\( a) \) A constant \( C_6 \) can be taken such that

\[
|b_k| \leq C_6 [1/d_1 (2l+p)^{\frac{8}{3}} c_i^{\frac{1}{k}}] J e^{-\frac{h}{s} + \frac{1}{2k}}, \quad J > 0
\]

implies \( u(x) = u(\rho, s) \in A \), where

\[
u_i(s) = \sum_{l=0}^{\infty} (2l+p)(\rho/r)^{i} u_i(s),
\]

\[
u_i^2(s) = \sum_{k=1}^{d_1} b_k^2 y_k^2(s).
\]

**Proof.** Under the assumption on \( b_k \), from lemma we obtain

\[
||u_i(s)||' \leq C_6 \cdot C_7 \cdot J e^{-\frac{h}{s} + \frac{1}{2k}} / (2l+p). \quad \text{We have, in } G,
\]
| \( u(\rho, s) \) | \( \leq \sum_{\ell=0}^{\infty} (2l+p)(R/r)^{\ell} \| u_{\ell}(s) \| \) \( \leq C_6 \cdot C_1 \sum_{\ell=0}^{\infty} A e^{-\delta \ell} = C_6 \cdot C_1 \cdot A e^{-1} - e^{-\delta} \).

This can be made \( \leq C \), where \( C \) is independent of \( A \).

\( \beta \) In expansion (10), we have

\[ |b_\ell^k| \leq C_\ell c_\ell^k \| u \|. \tag{11} \]

This is a consequence of \( |b_\ell^k| \leq \| u_{\ell}(s) \| \)' and (3).

10. Now put \( A = h / \log 1/\varepsilon \) and fix \( n \) (how to take \( n \) will be shown later). The set of \( u(\rho, s) = \sum_{\ell=0}^{\infty} (2l+p)(\rho/r)^{\ell} u_{\ell}(s) \) is a 2\( \varepsilon \)-separated subset of \( A \), if \( u_{\ell}(s) = \sum_{k=1}^{d_{s_k}^\ell} (s_k^\ell + \sqrt{-1} s_k^\ell) 2sC_\ell c_\ell^k y_k(s) \) where \( s_k, s_k^\ell \) are integers which satisfy

\[ |s_k| \leq (1/\sqrt{2}) C_\ell [1/d_\ell (2l+p)^{3/2} c_\ell^{1/2}] e^{-(\chi + A)/2s} \cdot C_\ell \cdot c_\ell^k. \tag{12} \]

Now \( n \) is defined as the largest of natural numbers \( l \) that make right hand side of (12) not smaller than 1. Then \( n \) can be estimated as follows: \( n = \log 1/\varepsilon / h \log e + O(\log \log 1/\varepsilon) \).

If we put \( M_l^2 = 2[(1/\sqrt{2}) C_\ell [1/d_\ell (2l+p)^{3/2} c_\ell^{1/2}] e^{-(\chi + A)/2s} \cdot C_\ell \cdot c_\ell^k] + 1 \),

we get

\[ M(2\varepsilon, A) \geq \prod_{l=0}^{\infty} \prod_{k=1}^{d_{s_k}^l} M_l^2. \]

Hence

\[ C(2\varepsilon, A) \geq \sum_{l=0}^{\infty} 2d_\ell \log (C_\ell [1/d_\ell (2l+p)c_\ell \cdot \varepsilon] e^{-(\chi + A)/2}) \]

\[ = 4/(p+2)! (\log 1/\varepsilon)^{p+2} / (h \log e)^{p+1} + O((\log 1/\varepsilon)^{p+1} \log \log 1/\varepsilon). \]

References

