38. A Theorem of Bari on the Completeness of Orthonormal Systems

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The purpose of the present note is to give another proof of the following

**Theorem.** If \( \{\varphi_n\} \) is a complete orthonormal system of a Hilbert space, and if \( \{\psi_n\} \) is another orthonormal system such as

\[
\sum_{n=1}^{\infty} \|\varphi_n - \psi_n\|^2 < \infty,
\]

then \( \{\psi_n\} \) is complete too.

The theorem is established by Nina Bari in 1941, according to her obituary note. K. Iséki, in a note [2] published in these Proceedings, summarized several extensions of her theorem due to several authors. Recently, G. Birkhoff and G.-C. Rota [1] reproduced the theorem in a connection with the Sturm-Liouville expansions.

In Birkhoff-Rota's proof, the following lemma plays a central role:

**Lemma.** Under the hypothesis of the theorem, if \( m \) is a natural number such as

\[
\sum_{n=m+1}^{\infty} \|\varphi_n - \psi_n\|^2 < 1,
\]

then the sequence of vectors

\[
\varphi_1, \varphi_2, \ldots, \varphi_m, \psi_{m+1}, \psi_{m+2}, \ldots
\]

is complete in the sense that no non-zero vector is orthogonal to (3).

Birkhoff-Rota's proof of the lemma is a simple application of the Parseval relation. In the present note, we shall give an alternative proof basing on the invertibility of an operator \( U \) defined by

\[
Ux = \sum_{n=m}^{m} \alpha_n \varphi_n + \sum_{n=m+1}^{\infty} \alpha_n \psi_n \quad \text{for} \quad x = \sum_{n=1}^{\infty} \alpha_n \varphi_n.
\]

We can easily obtain that \( U \) is a bounded operator which satisfies

\[
\|I - U\|^2 \leq \|I - U\|^2 = \sum_{n=m}^{\infty} \|(I - U)\varphi_n\|^2 = \sum_{n=m+1}^{\infty} \|\varphi_n - \psi_n\|^2 < 1,
\]

since the uniform norm \( \|T\| \) of an operator \( T \) is not greater than the Schmidt norm \( \|T\|_s \) (e.g. [3]). Hence \( U \) has an inverse, so that \( U \) has the dense range which is spanned by (3). This shows that (3) is complete.

In the remainder of our proof, we shall employ a method inspired

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by the second half of Iséki's proof which is somewhat simpler than that of Birkhoff-Rota.

Let $F$ be the orthocomplement of the space spanned by $\psi_{m+1}, \psi_{m+2}, \ldots$ and $P$ the projection belonging to $F$. To conclude the proof of the theorem, it remains to show that $F$ is $m$-dimensional. If $f \in F$ is orthogonal to $P\psi_1, P\psi_2, \ldots, P\psi_m$, then we have

$$(f, \psi_i) = (Pf, \psi_i) = (f, P\psi_i) = 0, \quad \text{for } i=1,2,\ldots,m,$$

whence Lemma implies $f=0$ since $f$ is orthogonal to (3). Hence $F$ is spanned by $P\psi_1, P\psi_2, \ldots, P\psi_m$, so that the dimension of $F$ is at most $m$. On the other hand, the dimension of $F$ is not less than $m$ since $F$ contains $\psi_1, \psi_2, \ldots, \psi_m$. Therefore $F$ is exactly $m$-dimensional.

In the same manner, the theorem is still true if (1) is replaced by

$$(1') \quad \sum_{n=1}^{\infty} ||\psi_n - \psi_n|| < \infty,$$

which is also due to N. Bari.

References