60. On Almost Periodic Transformations on Metric Space over Topological Semifield

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A. Edrei, P. Erdős, W. H. Gottschalk, G. A. Hedlund, and A. H. Stone have obtained interesting results on transformations on topological spaces (for references, see [5]). In this note, we shall consider some results in a metric space over a topological semifield (for related concepts, see [1] and [2]). Let $X$ be a metric space over a topological semifield $R$. We denote the metric by $\rho$. Let $f$ be a continuous mapping on $X$, i.e. $f(X) \subset X$.

We first repeat some definitions needed.

The mapping $f$ is said to be strongly almost periodic if for a given neighborhood $U$ there is a positive integer $k$ such that every $k$ consecutive positive integers contains an $n$ satisfying $\rho(x, f^n(x)) \in U$ for all $x \in X$.

In the definition of strongly almost periodicity, the positive integer $k$ is independently taken for each point $x$ of $X$. If $k$ depends on each point $x$, we need a new definition.

A point $x$ of $X$ is said to be almost periodic under $f$ (by W. H. Gottschalk [4]) if for a given neighborhood $U$, there is a positive integer $k$ such that every $k$ consecutive positive integers contains an $n$ satisfying $\rho(x, f^n(x)) \in U$. If each point $x$ is almost periodic under $f$, the mapping $f$ is said to be pointwise almost periodic. For $x \in X$, the set $\bigcup_{n=0}^{\infty} f^n(x)$ is called the orbit of $x$ under $f$ and the set $\bigcup_{n=-\infty}^{0} f^n(x)$ is called the semi-orbit of $x$ under $f$.

Under these concepts, we shall prove the following theorem which is formulated by P. Erdős and A. H. Stone [3].

Theorem. Let $X$ be a totally bounded metric space over a topological semifield, and $f$ a homeomorphism of $X$. If the set of all negative powers is equiuniformly continuous, then $f$ is strongly almost periodic.

The proof is quite similar with that of Theorem III of P. Erdős and A. H. Stone [3].

To prove Theorem, we take a neighborhood $U$ of $0$ in $R$. Then there is a neighborhood $W$ such that $W + W \subset U$. For $W$, there is a neighborhood $V$ of $0$ such that $\rho(f^{-n}(x), f^{-m}(y)) \in W$ holds for $x, y$ of $f^n(x)$ for which $\rho(x, y) \in V$. Here we can take $V$ and $W$ as saturated neighborhoods and $V \subset W$. 

Since $X$ is totally bounded, there is a finite partition $X_i(i = 1, 2, \cdots, r)$ of $X$ such that each $X_i$ is set of diameter less than $W$. For each $m$, we correspond a matrix of type $r \times r$: $A_m = (a_{ij}(m))$, where

$$a_{ij}(m) = \begin{cases} 1, & \text{if } f^m(X_i) \cap X_j \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, we have only finite distinct matrices. Let us suppose these matrices $A_1, A_2, \cdots, A_k$.

Let $m$ be a positive integer, then there is a positive integer $p$ such that $A_{m+k} = A_p$ and $p \leq k$. If $n$ is taken as $m + k - p$, then we have $m \leq n < m + k$. Next we shall show that $\rho(x, f^n(x)) \in U$ for all $x \in X$.

For $x$ of $X$, there are $X_i, X_j$ for which $x \in X_i, f^{n+k}(x) \in X_j$. Hence $a_{ij}(m+k) = 1$, and so we have $a_{ij}(p) = 1$. This shows $f^r(X_i) \cap X_j \neq 0$. If $y \in f^s(X_i) \cap X_j$, then we have

$$\rho(x, f^s(x)) \leq \rho(x, f^{s-p}(y)) + \rho(f^{s-p}(y), f^s(x))$$

$$= \rho(x, f^{s-p}(y)) + \rho(f^{s-p}(y), f^{s+k-p}(x))$$

$$\leq \delta(X_i) + \delta(f^{s-p}(X_i)) \in V + W \subset W + W \subset U,$$

where $\delta(A)$ is the diameter of $A$. Therefore $f$ is strongly almost periodic.

Next we shall consider a fundamental theorem on orbits.

Theorem. If $X$ is a metric space over a topological semifield $R$, and $x$ of $X$ is almost periodic under $f$, then the closure $Y$ of the orbit of $x$ under $f$ is minimal under $f$.

Remark. $Y$ of $X$ is said to be minimal under $f$ if $f(Y) = Y$ and $Y$ does not contain a proper closed and invariant subset under $f$.

To prove it, we suppose that $Y$ contains a proper closed and invariant subset $Z$. Then $Z$ does not contain $x$, and $Z$ is closed, hence there is a neighborhood $\mathcal{O}(x, U)$ of $x$ which does not meet $Z$, and further there is a saturated neighborhood $W$ of $0$ such that $W + W \subset U$. For $W$, we can find a positive integer $k$ such that every $k$ consecutive positive integers contains $n$ for which $\rho(x, f^n(x)) \in W$.

Further, for an element $z$ of $Z$ we can find a neighborhood $V$ such that $\rho(z, y) \in V$ implies $\rho(f^i(z), f^i(y)) \in W$ for $i = 1, 2, \cdots, k$. For $V$, there is a positive integer $p$ such that $\rho(z, f^p(x)) \in V$, from $x \in Y$. Therefore we have

$$\rho(f^i(z), f^{p+i}(x)) \in W.$$ 

On the other hand, we can find a positive integer $q$ such that $1 \leq q \leq k$ and $\rho(x, f^{p+q}(x)) \in W$. Hence we have

$$\rho(x, f^q(x)) \leq \rho(x, f^{p+q}(x)) + \rho(f^{p+q}(x), f^q(x)) \in W + W.$$ 

This shows $f^q(z) \in \mathcal{O}(x, W + W) \subset \mathcal{O}(x, U)$. $Z$ is invariant, so we have $f^q(z) \subset Z$. 
We have a similar theorem on semi-orbits. To do so, we must replace minimal into semi-minimal. The exact definition of the semi-minimal set is: a subset \( Y \) of \( X \) is said to be semi-minimal under \( f \), if the closure of semi-orbit of each \( y \) of \( Y \) is always \( Y \).

**Theorem.** If \( X \) is a metric space over a topological semifield, and \( x \) of \( X \) is almost periodic under \( f \), the closure \( Y \) of the semi-orbit of \( x \) under \( f \) is semi-minimal.

We have not any difficulty in the proof. Suppose that \( Y \) is not semi-minimal, then there is a point \( y \) of \( Y \) such that the closure of the semi-orbit of \( y \) is not \( Y \). Further the closure \( Z \) of the semi-orbit of \( y \) does not contain \( x \), i.e. \( x \notin Z \). Hence, we can find a neighborhood \( \Omega(x, U) \) of \( x \) which does not meet \( Z \). Next we take a saturated neighborhood \( W \) such that \( W + W \subseteq U \). For \( W \), we can find a positive integer \( k \) such that every set of \( k \) consecutive positive integers contains \( n \) for which \( \rho(x, f^n(x)) \in W \). Since the mapping \( f \) is continuous, we can take a neighborhood \( V \) of \( 0 \) in \( R \) such that \( \rho(x, x') \in V \) implies \( \rho(f^i(x), f^i(x')) \in W \) for \( i = 1, 2, \ldots, k \), where \( x' \in X \).

\( Y \) is the closure of the semi-orbit of \( x \), so we take a \( p \) integer \( p \) such that \( \rho(y, f^p(x)) \in V \). For \( p \), we can find an integer \( q \) such that \( 1 \leq q \leq k \) and \( \rho(x, f^{p+q}(x)) \in W \). Therefore we have

\[
\rho(x, f^q(y)) < \rho(x, f^{p+q}(x)) + \rho(f^{p+q}(x), f^q(y)) \in W + W.
\]

Hence we have \( f^q(y) \in \Omega(x, U) \), which is impossible.

Consequently, we have the following theorem from the above two results.

**Theorem.** If \( X \) is a metric space over a topological semifield, and \( f \) is pointwise almost periodic, then \( f \) gives an orbit (or a semi-orbit) closure decomposition.

**References**