61. On Linear Isotropy Group of a Riemannian Manifold

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Introduction. Let $M$ be a connected Riemannian manifold of dimension $n$ and of class $C^\infty$, and let $M_p$ be the tangent space of $M$ at $p$. According to the Riemannian structure a scalar product $g_p(X, Y)$ is defined for any vectors $X, Y \in M_p$. We denote by $L_p$ the group of all linear transformations of $M_p$. The infinitesimal linear isotropy group $K_p$ is, by definition [2], the subgroup of $L_p$ consisting of all linear transformations of $M_p$ which leave invariant the curvature tensor $R$ and the successive covariant differentials $\nabla R$, $\nabla^2 R$, $\cdots$ at $p$. We define a group $A_p$ as a subgroup of $K_p$ consisting of all elements of $K_p$ which leave invariant the scalar product $g_p(X, Y)$. Let $I(M)$ be the group of isometries of $M$. Let $H_p$ be the isotropy group of $I(M)$ at $p$, and let $dH_p$ be the linear isotropy group of $H_p$. In §1, we shall investigate sufficient conditions that $dH_p = A_p$. §2 is devoted to applications of the main theorem to Riemannian globally symmetric spaces.

§1. Main theorem.

Theorem 1. If $M$ is a simply connected homogeneous Riemannian manifold, then $dH_p = A_p$ for each $p$ in $M$.

In order to prove this theorem, we need the following:

Lemma. If $M$ is an analytic complete simply connected Riemannian manifold, then $dH_p = A_p$ for each $p$ in $M$.

Proof. We have proved that $dH_p \subset A_p$ for any Riemannian manifold [3] p. 1). Take a normal coordinate system $\{x_1, \cdots, x_n\}$ at $p$, with coordinate neighborhood $U$. We may assume that $\{(\partial/\partial x_i)_p, \cdots, (\partial/\partial x_n)_p\}$ is an orthonormal base, and that $U$ is the interior of a geodesic sphere centered at $p$. $U$ has the Riemannian metric induced from $M$. Since $M$ is analytic, each element $a \in A_p$ induces a local isometry $\bar{f}$ which maps $U$ onto itself, such that $\bar{f}(p) = p$ and $(df)_p = a$ ([3] p. 2). Since $M$ is a simply connected complete analytic Riemannian manifold, and $U$ is a connected open subset of $M$, this local isometry $\bar{f}$ can be uniquely extended to $f$, an isometry of $M$ ([4] p. 256). Clearly $f(p) = p$ and $(df)_p = a$. Therefore we have $A_p \subset dH_p$.

Proof of Theorem. Since $M$ is a Riemannian homogeneous
space of a Lie group, it can be considered to be an analytic complete Riemannian manifold. Since $M$ is simply connected it satisfies the conditions of the lemma.

Counterexample. Consider in $E^3$ a cylinder of revolution with the natural Riemannian metric from Euclidean metric in $M$. This is a homogeneous Riemannian manifold, which is not simply connected. In this case, $dH_p = \text{identity}$ and $A_p$ is the rotation group of $E^3$. This example shows that the simply connectedness of the theorem cannot be removed.

Corollary. If $M$ is an analytic complete simply connected Riemannian manifold, then $H_p$ is isomorphic to $dH_p$ as Lie groups.

Proof. Let $U$ be the neighborhood with the same Riemannian structure as in above lemma. Let $\tilde{H}_p$ be the group of all isometries of $U$ which fix the point $p$. Then each element $f \in H_p$ induces $f \mid_U \in \tilde{H}_p$. Since $M$ is a simply connected analytic complete Riemannian manifold, each $\tilde{f} \in \tilde{H}_p$ can be extended uniquely to $f$, an isometry of $M$. Clearly $f \mid_U = \tilde{f}$. Therefore $H_p$ is isomorphic to $\tilde{H}_p$ as Lie groups. Each element of $A_p$ can be expressed by a matrix with respect to the base $\{ \partial / \partial x_1, \ldots, \partial / \partial x_n \}$. In this coordinate system $\{ x_1, \ldots, x_n \}$ each element of $\tilde{H}_p$ can be expressed by

$$y_i = \sum_{j=1}^n a_{ij} x_j (i = 1, 2, \ldots, n),$$

where the matrix $|a_{ij}|$ belongs to $A_p$([3] p. 3). This means that $\tilde{H}_p$ is isomorphic to $A_p$ as Lie groups. But $A_p = dH_p$. Therefore $H_p$ is isomorphic to $dH_p$, and this isomorphism is given by the correspondence $f \in H_p \rightarrow (df)_p \in dH_p$.

§ 2. Applications.

In 1927, E. Cartan proved the following theorem ([1] p. 84).

Let $M$ be an affine locally symmetric space without torsion. If a linear transformation of $M_p$ leaves invariant the curvature tensor $R$ at $p$, then this induces a local affine isomorphism on $M$.

We shall treat this problem globally imposing some conditions on $M$.

Theorem 2. If $M$ is a simply connected Riemannian globally symmetric space, then $dH_p = A_p$.

Proof. $M$ is a simply connected homogeneous Riemannian manifold. Since $M$ is locally symmetric, the tensors $\nabla^k R$ vanish for $k = 1, 2, \ldots$. By Theorem 1 the conclusion follows.

A Riemannian globally symmetric space $M$ is said to be of the non-compact type, if the Riemannian symmetric pair $(G, K)$ is of the noncompact type ([5] p. 194), where $G$ is the identity component of $I(M)$ and $K$ is the isotropy group of $G$ at some point in $M$. Let us fix a point $p$ and let $A$ the space of $A_p$. 

Theorem 3. If $M$ is a Riemannian globally symmetric space of the noncompact type, then the space of $I(M)$ is diffeomorphic to $E^n \times A$.

Proof. For a Riemannian symmetric space of the noncompact type $M$, there is a normal coordinate system whose coordinate neighborhood is $M$ ([5] p. 215). This means that $M$ is diffeomorphic to $E^n$. Let $0(M)$ be the bundle of orthonormal frames over $M$, and let $F$ be an orthonormal frame at $p$. For each $q \in M(q \neq p)$ we put $f_q = \tau_{qp}F$ where $\tau_{qp}$ is the parallel translation along the unique geodesic segment from $p$ to $q$. Therefore we get a $C^\infty$ cross-section in the principal bundle $0(M)$, so that this bundle is equivalent to a product bundle. Each member of $I(M)$ induces a diffeomorphism on $0(M)$ in the natural way. Then the set of frames $I(M)F$ can be considered as a reduced bundle of $0(M)$. Clearly the bundle $I(M)F$ is equivalent to a product bundle. In this bundle, the base space is diffeomorphic to $E^n$, and the standard fiber is diffeomorphic to $A$.

Corollary. If $M$ is a Riemannian globally symmetric space of the non-compact type, then $dH_p = A_p$.

Proof. Since $M$ is simply connected, by Theorem 2 the conclusion follows.

References