1.1. Definition. Let \( \sum_{n=0}^{\infty} a_n \) be a given infinite series and \( s_n^\alpha \) be the \( n \)-th Cesàro mean of order \( \alpha \) of the sequence \( \{s_n\} \), where \( s_n \) is the partial sum of the given series. We say that the series \( \sum_{n=0}^{\infty} a_n \) is absolutely summable \((C, \alpha)\), or summable \(|C, \alpha|\), if the series \( \sum_{n=1}^{\infty} |s_n^\alpha - s_{n-1}^\alpha| \) converges.

A sequence \( \{\lambda_n\} \) is said to be convex when \( \lambda_{2n} \geq 0 \) \((n = 1, 2, \cdots)\), where \( \lambda_{2n} = \lambda_n - \lambda_{n+1}, \lambda_{2n+1} = \lambda_{2n+2} - \lambda_{2n+1} \). It is known\(^1\) that if \( \{\lambda_n\} \) is a convex sequence and the series \( \sum_{n=1}^{\infty} n^{-1} \lambda_n \) converges, then \( \lambda_n \) is non-negative and non-increasing.

1.2. Let \( f(t) \) be a periodic function with period \( 2\pi \), and integrable in the sense of Lebesgue over \((-\pi, \pi)\). Without any loss of generality we may assume that the constant term in the Fourier series of \( f(t) \) is zero, so that

\[
f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t),
\]

where \( A_n(x) = a_n \cos nx + b_n \sin nx \). Let us put

\[
S_n(x) = \sum_{v=1}^{\infty} A_v(x), \quad D_n(t) = \frac{1}{2} \sum_{v=1}^{n} \cos nt = \frac{\sin \left( n + \frac{1}{2} \right) t}{2 \sin \frac{t}{2}}
\]

and \( \phi(t) = \phi_x(t) = \frac{1}{2} \{f(x + t) + f(x - t) - 2f(x)\} \).

1.3. Recently, Pati has proved the following result:

Theorem A.\(^2\) If \( \{\lambda_n\} \) is a convex sequence such that \( \sum_{n=1}^{\infty} n^{-1} \lambda_n (\log n)^{\frac{1}{2}} < \infty \), then \( \sum_{n=1}^{\infty} \lambda_n A_n(t) \) is summable \(|C, 1|\) at every point \( t = x \) at which

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\(^1\) H. C. Chow [1], Lemma 4.
\(^2\) T. Pati [2], Theorem 2.
2. The object of this paper is to prove the following two theorems:

Theorem 1. If \( \{\lambda_n\} \) is a convex sequence such that

\[
\frac{\sum_{n=1}^{\infty} \lambda_n (\log n)^{\frac{d}{2} (1-\alpha)}}{n} < \infty \quad (0 \leq \alpha < 1),
\]

then \( \sum_{n=1}^{\infty} \lambda_n A_n(t) \) is summable \(| C, 1 |\) at every point \( t = x \) at which

\[
\Phi(t) = \int_{t}^{\infty} |\phi(u)| \, du = o\left(\frac{t}{\log \frac{1}{t}}\right), \quad \text{as} \quad t \to 0.
\]

In the case \( \alpha = 0 \), we get Theorem A. In the limiting case, \( \alpha = 1 \), we have the following

Theorem 2. If \( \{\lambda_n\} \) is a convex sequence such that

\[
\sum_{n=1}^{\infty} \lambda_n (\log \log n)^{\frac{d}{2}} < \infty,
\]

then \( \sum_{n=1}^{\infty} \lambda_n A_n(t) \) is summable \(| C, 1 |\) at every point \( t = x \) at which

\[
\Phi(t) = \int_{t}^{\infty} |\phi(u)| \, du = o\left(\frac{t}{\log \frac{1}{t}}\right), \quad \text{as} \quad t \to 0.
\]

3. Proof of Theorem 1. We require the following lemmas:

Lemma 1. If \( \{\lambda_n\} \) is a convex sequence such that \( \sum_{n=1}^{\infty} n^{-1} \lambda_n < \infty \) and (1.1) holds, then \( \sum_{n=1}^{\infty} \lambda_n A_n(x) \) is summable \(| C, 1 |\) if and only if

\[
\sum_{n=1}^{\infty} n^{-1} |s_n(x) - f(x)| < \infty.
\]

Lemma 2. If (2.2) holds, then

\[
\sum_{v=1}^{n} (s_v(x) - f(x))^2 = o(n(\log n)^{1-\alpha}),
\]

and further, by Cauchy’s inequality, we have

\[
\sum_{v=1}^{n} |s_v(x) - f(x)| = o(n(\log n)^{\frac{d}{2} (1-\alpha)}), \quad \text{as} \quad n \to \infty.
\]

Proof. First, under the condition (2.2), we shall estimate the order of the integrals \( \int_{t}^{\infty} |\phi(t)| \, dt \) and \( \int_{t}^{\infty} |\phi(t)| \, dt \). By integration by parts, we get

\[
\int_{t}^{\infty} |\phi(t)| \, dt = \left[ \phi(t) \right]_{t}^{\infty} + \int_{t}^{\infty} \phi(t) \, dt
\]

3) T. Pati [2], Theorem 1.
\[
\begin{align*}
&= O(1) + o\left(\frac{1}{(\log n)^a}\right) + o\left(\int_1^n \frac{1}{t^a (\log \frac{1}{t})^b} \, dt\right) = o((\log n)^{-a}). \\
\text{(3.2)} & \quad \int_1^n \frac{\phi(t)}{t^a} \, dt = \left[ \frac{\Phi(t)}{t^a} \right]_1^n + 2 \int_1^n \frac{\phi(t)}{t^a} \, dt \\
&= O(1) + o\left(\frac{n}{(\log n)^a}\right) + o\left(\int_1^n \frac{1}{t^a (\log \frac{1}{t})^b} \, dt\right) = o\left(\frac{n}{(\log n)^a}\right).
\end{align*}
\]

Now
\[
\begin{align*}
&\sum_{v=1}^{n} [s(x) - f(x)]^2 = \sum_{v=1}^{n} \left(\frac{2}{n} \int_0^n \phi(t) \sin vt \, dt + o(1)\right)^2 \\
&= \sum_{v=1}^{n} \left\{ 4 \int_0^n \phi(t) \sin vt \, dt \int_0^n \phi(u) \sin vu \, du + o\left(\int_0^n \phi(t) \sin vt \, dt + o(1)\right)\right\} \\
&= 4 \int_0^n \phi(t) \, dt \int_0^n \phi(u) \left(\sum_{v=1}^{n} \sin vt \sin vu\right) du + o\left(\sum_{v=1}^{n} \int_0^n \phi(t) \sin vt \, dt + o(n)\right) \\
&= I_1 + o\left(\sqrt{I_1}\right) + o(n),
\end{align*}
\]

where
\[
I_1 = 4 \int_0^n \phi(t) \, dt \int_0^n \phi(u) \left(\sum_{v=1}^{n} \sin vt \sin vu\right) du.
\]

We shall devide \( I_1 \) into four parts
\[
I_1 = \frac{1}{\pi} \int_0^n \phi(t) \, dt \left(\sum_{v=1}^{n} \sin vt \sin vu\right) du.
\]

By condition (2.2), we get
\[
|J_1| \leq \frac{4}{\pi^2} \int_0^n |\phi(t)| \, dt \int_0^n |\phi(u)| \left(\sum_{v=1}^{n} v^2\right) du = o\left(\frac{n}{(\log n)^{2a}}\right).
\]

By (2.2) and (3.1), we get
\[
|J_2| \leq \frac{4}{\pi^2} \int_0^n |\phi(t)| \, dt \int_0^n |\phi(u)| \left(\sum_{v=1}^{n} v^2\right) du = o\left(\frac{1}{n(\log n)^a(\log n)^{-a} n^2}\right)
\]

and
\[
|J_3| = o\left(\frac{n(\log n)^{-a}}{\log n^a}\right).
\]

\( J_3 \) is equal to \( J_2 \). Hence it remains to estimate \( J_4 \):
\[
J_4 = \frac{2}{\pi^2} \int_0^n \phi(t) \, dt \int_0^n \phi(u) \left(\sum_{v=1}^{n} (\cos v(u-t) - \cos v(u+t))\right) du \\
= \frac{2}{\pi^2} \int_0^n \phi(t) \, dt \int_0^n \phi(u) \left(D_v(u-t) - D_v(u+t)\right) du \\
= o\left(\int_0^n |\phi(t)| \, dt \int_0^n |\phi(u)| \left|\frac{\sin \left(n + \frac{1}{2}\right)(u-t)}{|u-t|}\right| du\right).
\]
\[ J'_1 = o \left( \int_{\pi}^{\pi} \left| \phi(t) \right| dt \int_{\pi}^{\pi} \left| \phi(u) \right| \sin \left( n + \frac{1}{2} \right) \left| u - t \right| du \right) = O(J'_4 + J''_4). \]

Then
\[ J'_1 = \int_{\pi}^{\pi} \left| \phi(t) \right| dt \left\{ \int_{\pi}^{\pi} \left| \phi(u) \right| \sin \left( n + \frac{1}{2} \right) \left| u - t \right| du \right\} \]

By integration by parts and by (2.2) and (3.1), we get
\[ J'_1 \leq \left( n + \frac{1}{2} \right) \int_{\pi}^{\pi} \left| \phi(t) \right| dt \int_{\pi}^{\pi} \left| \phi(u) \right| du \]

By integration by parts and by (2.2) and (3.1), we get
\[ J'_1 \leq \left( n + \frac{1}{2} \right) \int_{\pi}^{\pi} \left| \phi(t) \right| dt \int_{\pi}^{\pi} \left| \phi(u) \right| du \]

By integration by parts and by (2.2) and (3.1), we get
\[ J'_1 \leq \left( n + \frac{1}{2} \right) \int_{\pi}^{\pi} \left| \phi(t) \right| dt \int_{\pi}^{\pi} \left| \phi(u) \right| du \]

By integration by parts and by (2.2) and (3.1), we get
\[ J'_1 \leq \left( n + \frac{1}{2} \right) \int_{\pi}^{\pi} \left| \phi(t) \right| dt \int_{\pi}^{\pi} \left| \phi(u) \right| du \]

By integration by parts and by (2.2) and (3.1), we get
\[ J'_1 \leq \left( n + \frac{1}{2} \right) \int_{\pi}^{\pi} \left| \phi(t) \right| dt \int_{\pi}^{\pi} \left| \phi(u) \right| du \]

Thus we get the conclusion
\[ \sum_{v=1}^{n} (s_v(x) - f(x))^2 = o(n(\log n)^{1-\alpha}). \]
Lemma 3.\textsuperscript{4}) If \( \{ \lambda_n \} \) is a convex sequence such that \( \sum_{n=1}^{\infty} n^{-1} \lambda_n < \infty \), then
\[
\sum_{n=1}^{m} \log (n+1) \lambda_n = O(1),
\]
as \( m \to \infty \), and \( \lambda_m \log m = o(1) \), as \( m \to \infty \).

We shall now proceed to prove Theorem 1.

By Lemma 1, it is enough to prove that
\[
\sum_{n=1}^{\infty} n^{-1} \lambda_n \left| s_n(x) - f(x) \right| < \infty.
\]

By Abel's transformation,\textsuperscript{5)}
\[
\sum_{v=1}^{n} v^{-1} \lambda_v \left| s_v(x) - f(x) \right| = n^{-1} \lambda_n \sum_{v=1}^{n} \left| s_v(x) - f(x) \right| + \frac{1}{v(v+1)} \sum_{v=1}^{n-1} \lambda_v \sum_{\mu=1}^{v} \left| s_\mu(x) - f(x) \right| + o(\lambda_n (\log n)^{\frac{1}{2}}) + o\left( \sum_{v=1}^{n} v^{-1} \lambda_v (\log v)^{\frac{1}{2}} \right),
\]
by Lemma 2, and then, by our hypothesis and Lemma 3,
\[
\sum_{v=1}^{n} v^{-1} \lambda_v \left| s_v(x) - f(x) \right| = o(1),
\]
as \( n \to \infty \).

4. Proof of Theorem 2. For the proof of Theorem 2, we need Lemma 1, Lemma 3, and the following

Lemma 4. If (2.4) holds, then
\[
\sum_{v=1}^{n} (s_v(x) - f(x))^2 = o(n \log \log n),
\]
and further, by Cauchy's inequality, we have
\[
\sum_{v=1}^{n} \left| s_v(x) - f(x) \right| = o(n (\log \log n)^{\frac{1}{2}}), \quad \text{as} \quad n \to \infty.
\]

This lemma can be proved by the same idea as in the proof of Lemma 2.

We shall now prove Theorem 2. By Lemma 1, it is sufficient to prove that
\[
\sum_{n=1}^{\infty} n^{-1} \lambda_n \left| s_n(x) - f(x) \right| < \infty.
\]

By Abel's transformation, we have
\[
\sum_{v=1}^{n} v^{-1} \lambda_v \left| s_v(x) - f(x) \right|.
\]

\textsuperscript{4) T. Pati [2], Lemma 3.}
\textsuperscript{5) Cf. Pati [2].}
\[ n^{-1} \sum_{\mu=1}^{\nu} |s_{\mu}(x) - f(x)| + n^{-1} \sum_{\nu=1}^{\nu} |s_{\nu}(x) - f(x)| \]

\[ = n^{-1} \sum_{\nu=1}^{\nu} \frac{\lambda_{\nu}}{v(v+1)} |s_{\nu}(x) - f(x)| + \sum_{\nu=1}^{\nu} \frac{A_{\nu}}{v+1} \sum_{\mu=1}^{\nu} |s_{\mu}(x) - f(x)| \]

\[ + o(\lambda_{\nu} (\log \log \nu)^{3/2}) \quad (\text{by lemma 4}) \]

\[ = o(1), \]

as \( n \to \infty \), by our hypothesis and Lemma 3.

References
