71. On Holomorphic Markov Processes

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Under appropriate regularity conditions, a temporally homogeneous Markov process is associated with a contraction semi-group \( \{T_t; t \geq 0\} \) of class \((C_0)\) \([1]\) in a suitable Banach space \(X\). In certain cases where \(X\) are complex Banach spaces, \(T_t\) admits a holomorphic extension \(T_\lambda\) given by strongly convergent Taylor series for all \(x \in X\):

\[
T_\lambda x = \sum_{n=0}^{\infty} \frac{(-\lambda t)^n}{n!} T_0^{[n]}x \quad \text{for} \quad \left| \frac{\lambda - t}{t} \right| \leq \text{some positive constant} \ C,
\]

the existence of the \(n\)-th strong derivative \(T_0^{[n]}x\) in \(x\) of \(T_t x\) being assumed for any \(t > 0\) and any \(x \in X\) \((n=1, 2, \ldots)\). Such is the case of the semi-group

\[
(T_0f)(x) = \int_{-\infty}^{\infty} e^{-ix-\frac{1}{2}t^2} f(y) dy \quad (t > 0),
\]

\[
= f(x) \quad (t = 0)
\]

in the Banach space \(C[-\infty, \infty]\) of bounded uniformly continuous, complex valued functions \(f(x)\) on \((-\infty, \infty)\) endowed with the maximum norm. Suggested by this example, we shall call a Markov process a **holomorphic Markov process** if the associated semi-group \(T_t\) admits a holomorphic extension \(T_\lambda\) of the form given in (1). This notion seems to be of some interest. For instance, we can prove

**Proposition.** Let a semi-group \(T_t\) with the infinitesimal generator \(A\) be associated with a holomorphic Markov process through

\[
(T_t f)(x) = \int P(t, x, dy) f(y), \quad f \in X
\]

where \(P(t, x, dy)\) is the transition probability of this process. Suppose that \(T_{t_0}f_0 = 0\) for some \(t_0 > 0\) and \(f_0 \in X\). Then \(f_0 = 0\).

**Proof.** We have \(A^* T_{t_0} f_0 = T_{t_0}^{[1]} f_0 = 0\) \((n=0, 1, \ldots)\) by the linearity of \(A\). Hence, by Taylor expansion (1), we see that \(T_{t_0} f_0 = 0\) for \(\left| \lambda - t \right| /t \leq C\). Repeating the argument, we easily see that \(T_{t} f_0 = 0\) for all \(t > 0\) and so \(f_0 = \lim_{t \to 0} T_{t} f_0 = 0\).

There are abundant examples of holomorphic Markov processes. In fact, the fractional power \([2]\) \(A_\alpha\) \((0 < \alpha < 1)\) of the infinitesimal generator \(A\) of a contraction semi-group \(T_t\) of class \((C_0)\) generates a contraction semi-group \(\hat{T}_{t, \alpha}\) of class \((C_0)\) which admits a holomor-
The purpose of the present note is to devise another method for the construction of holomorphic Markov processes. It is based upon

**Theorem.** Let $B$ be the infinitesimal generator of an equi-continuous group of class $(C_0)$ in a complex Banach space $X$. Then $A = B^*$ is the infinitesimal generator of an equi-continuous semi-group of class $(C_0)$ which is also a holomorphic semi-group [3] characterized by any one of the following three conditions:

(I) For all $t > 0$, $T_t X \subseteq D(A)$, the domain of $A$, and there exists a positive constant $C_0$ such that the family of operators $\{(C, tT_t^n); 0 < t \leq 1, n = 0, 1, \ldots\}$ is equi-continuous.

(II) $T_t$ admits a holomorphic extension $T_\lambda$ of the form given in (1) such that the family of operators $\{e^{\lambda \tau T_\lambda^n}; |\arg \lambda| \leq \tan (k^{-1}C_0)\}$ with some fixed $k > 0$ is equi-continuous.

(III) There exists a positive constant $C_1$ such that the family of operators $\{(C_1(\lambda I - A)^{-n}); \Re (\lambda) \geq 1 \text{ and } n = 0, 1, \ldots\}$ is equi-continuous.

**Proof.** Since $B$ generates an equi-continuous group of class $(C_0)$, $D(B)$ is dense in $X$ and the resolvents $(\sqrt{\lambda} I \pm B)^{-1}$ both exist as bounded linear operators on $X$ into $X$ for $\Re (\sqrt{\lambda}) > 0$ satisfying the condition

(5) \[\{(\Re (\sqrt{\lambda})(\sqrt{\lambda} I \pm B)^{-1})^n; \Re (\sqrt{\lambda}) > 0 \text{ and } n = 0, 1, \ldots\}\]

is equi-continuous.

Thus, by

(6) \[-(\lambda I - A)^{-1} = (\sqrt{\lambda} I - B)^{-1}(\sqrt{\lambda} I + B)^{-1}(\Re (\lambda) > 0)\]

we see that $D(A) = \text{the range of } (\lambda I - A)^{-1}$ is dense in $X$ with $D(B)$. (6) implies also that

(7) \[\{(\lambda(\lambda I - A)^{-1})^n = (\sqrt{\lambda}(\sqrt{\lambda} I - B)^{-1} \cdot \sqrt{\lambda}(\sqrt{\lambda} I + B)^{-1})^n; \lambda > 0, n = 0, 1, 2, \ldots\}\]

is equi-continuous.

Hence $A$ generates an equi-continuous semi-group of class $(C_0)$. Moreover,

\[\left\{\left(\left(\sqrt{1 + \tau^2} \cos \left(\frac{1}{2} \tan^{-1} \tau\right)(1 + i\tau I - A)^{-1}\right)\right)^n\right\}\]

\[= \{(\Re (\sqrt{1 + i\tau})(\sqrt{1 + i\tau} I - B)^{-1} \Re (\sqrt{1 + i\tau})(\sqrt{1 + i\tau} I + B)^{-1})^n\}\]
is equi-continuous in $-\infty < \tau < \infty$ and in $n=0, 1, \ldots$. Hence, by (III), the operator $A$ generates a holomorphic semi-group.

An example of holomorphic Markov processes. Let $X=C[-\infty, \infty]$ and consider the operator

$$ A = a^2(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx} + q(x). $$

Suppose that $a(x), a'(x), b(x)$, and $q(x)$ are uniformly continuous, bounded real-valued functions in $(-\infty, \infty)$ satisfying conditions

(9) $q(x) \leq 0$ and $0 < \delta \leq a(x)$ in $(-\infty, \infty)$, where $\delta$ is a positive constant.

Then $A$ generates a contraction holomorphic semi-group $T_t$ in $X$ which is positive, i.e., $f(x) \geq 0$ in $(-\infty, \infty)$ implies $(T_t f)(x) \geq 0$ in $(-\infty, \infty)$. Thus $T_t$ is associated with a holomorphic Markov process.

Proof. A may be written as

$$ A' = \left( a(x) \frac{d}{dx} \right)^2 + p(x) \frac{d}{dx} - \varepsilon \frac{d}{dx} + q(x), $$

where

$$ p(x) = b(x) - a(x) a'(x) + \varepsilon \sup_{-\infty < x < \infty} \left| b(x) - a(x) a'(x) \right|. $$

We shall prove (i): $E = \left( a(x) \frac{d}{dx} \right)^2$ generates a contraction positive holomorphic semi-group in $X$, (ii): $p \frac{d}{dx}$ and $-\varepsilon \frac{d}{dx}$ both generate contraction positive semi-groups of class $(C_0)$ in $X$, (iii): $q(x)$ generates a contraction positive semi-group of class $(C_0)$ in $X$, (iv): for $1 > \alpha > \frac{1}{2}$, the domain $D\left( p \frac{d}{dx} \right)$ contains the domain $D(\hat{E}_\alpha)$, where $\hat{E}_\alpha$ is the fractional power operator of $E$ and (v): for $1 > \alpha > \frac{1}{2}$, the domain $D\left( -\varepsilon \frac{d}{dx} \right)$ contains the domain $D(\hat{F}_\alpha)$, where $\hat{F}_\alpha$ is the fractional power operator of $F = E + p \frac{d}{dx}$ with the domain $D(F) = D(E)$.

Then, by a theorem proved in a preceding note in these Proceedings [4], $F = \left( E + p \frac{d}{dx} \right)$ generates a contraction holomorphic semi-group in $X$. By H. F. Trotter’s product formula [5], we have

$$ e^{t(\varepsilon + p \frac{d}{dx})} = \lim_{n \to \infty} \left( e^{t\frac{d}{dx}} \right)^n $$

so that the semi-group $e^{t(\varepsilon + p \frac{d}{dx})}$ generated by $F$ is positive by the positivity of semi-groups $e^{t\frac{d}{dx}}$. Similarly, by (v), $F - \varepsilon \frac{d}{dx}$ with the domain $D\left( F - \varepsilon \frac{d}{dx} \right) = D(F)$ generates a positive contrac-
tion holomorphic semi-group in $X$. The multiplication operator $q$ is a bounded operator in $X$ which generates a positive contraction semi-group of class $(C_0)$ in $X$ by (iii). Hence, by a similar argument as above, $A=F-\varepsilon \frac{d}{dx}+q$ with the domain $D(A)=D(F)$ generates a positive contraction holomorphic semi-group in $X$.

The proof of (i) through (iv) is given as follows.

(i): $B=a \frac{d}{dx}$ generates a positive contraction group of class $(C_0)$ in $X$ of translations

$$f(x(y)) \rightarrow f(x(y\pm t)),$$

where $y(x)=\int_0^x a(s)^{-1}ds$.

Hence the resolvents $(\sqrt{\lambda} I \mp B)^{-1}$ are positive operators in $X$ for $\lambda>0$ and so $(\lambda I-E)^{-1}=(\sqrt{\lambda} I-B)^{-1}(\sqrt{\lambda} I+B)^{-1}$ is a positive operator in $X$. Thus, remembering the Theorem and the representation

$$e^{tE}=s\lim_{n\to\infty} \left( I-\frac{t}{n} E \right)^{-n},$$

we have proved (i).

(ii): As in (i), we prove that $p \frac{d}{dx}$ and $-\varepsilon \frac{d}{dx}$ both generate positive contraction group of class $(C_0)$ in $X$.

(iv): The resolvent of $\hat{E}_a$ is given by T. Kato’s formula [6]

$$\frac{1}{\lambda I-E} = \frac{1}{(\sqrt{\lambda} I-B)^{-1}} \frac{1}{(\sqrt{\lambda} I+B)^{-1}}.$$}

We have

$$B(\lambda I-E)^{-1} = B(\sqrt{\lambda} I-B)^{-1}(\sqrt{\lambda} I+B)^{-1}$$

and so, by $\| (\sqrt{\lambda} I+B)^{-1} \| \leq r^{-1/2}$, we see that the right side of

$$B(\lambda I-\hat{E}_a)^{-1} = \sin \frac{\alpha \pi}{\pi} \int_0^\pi (\lambda I-E)^{-1} \frac{r^a}{r^2-2\lambda r^a \cos \alpha \pi + r^{2a}} dr$$

converges when $1>\alpha>\frac{1}{2}$. This proves (iv).

(v): Remembering

$$B(rI-F)^{-1} = B(I-F)^{-1}(I-F)(rI-F)^{-1}$$

and

$$B(rI-F)^{-1} = B(rI-E)^{-1}\left\{ I-\frac{d}{dx}(rI-E)^{-1} \right\}^{-1}$$

we prove (v) as in (iv).

Remark. That $b(x)$ in (8) may change sign on $(-\infty, \infty)$ was suggested, thanks to a conversation with Professor S. Ito and Dr. H. Tanaka.
References


