98. On Kernels of Invariant Functional Spaces
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Introduction. Deny introduced in [6] the notion of invariant functional spaces and he proved that to an invariant functional space $\mathcal{F}$ corresponds a convolution kernel $\kappa$ in the following sense: each potential $u_f$ in $\mathcal{F}$ generated by a bounded measurable function $f$ with compact support is equal to the convolution $\kappa * f$. In this paper, we shall prove that the converse is valid. That is, for a positive measure $\kappa$ of positive type, there exists an invariant functional space with kernel $\kappa$. Furthermore, we shall give a necessary and sufficient condition for a positive measure $\kappa$ of positive type to be the kernel of a special Dirichlet space.

1. Invariant functional spaces. Let $X$ be a locally compact abelian group. We denote by $dx$ the Haar measure of $X$. We define two kinds of functional spaces on $X$.

Definition 1. A weak invariant functional space $\mathcal{F}=\mathcal{F}(X)$ with respect to $X$ and $dx$ is a Hilbert space of real-valued locally summable functions satisfying the following two conditions.

1.1 For any compact subset $K$ in $X$, there exists a positive constant $A(K)$ such that
$$\int_K u(x)dx \leq A(K)||u||$$
for any $u$ in $\mathcal{F}$.

1.2 Let $U_x u$ be a function obtained from $u$ in $\mathcal{F}$ by the translation $x \in X$. For any $u$ in $\mathcal{F}$ and any $x$ in $X$, $U_x u$ is in $\mathcal{F}$ and $||U_x u||=||u||$.

Two functions which are equal $p.p.$ in $X$ represent the same element in $\mathcal{F}$. By the condition (1.1), for any compact subset $K$ in $X$, there exists an element $u_K$ in $\mathcal{F}$ such that
$$(u, u_K) = \int_K u(x)dx$$
for any $u$ in $\mathcal{F}$. Especially when $u_K(x) \geq 0$ p.p. in $X$ for any compact subset $K$, $\mathcal{F}$ is called a positive weak invariant functional space on $X$.

Definition 2. A weak invariant functional space $\mathcal{F}$ is called

1) A property is said to hold p.p. in a subset $E$ in $X$ if the property holds in $E$ except a set which is locally of measure zero.
2) Cf. [6], p. 12.
an invariant functional space on $X$ if the following additional condition is satisfied.

(2.1) For any compact subset $K$ in $X$, there exists a positive constant $A(K)$ such that

$$\int_K |u(x)| dx \leq A(K) ||u||$$

for any $u$ in $\mathcal{F}$.

Let $\mathcal{F}$ be an invariant functional space on $X$. By the condition (2.1) in the above definition, for any bounded measurable function $f$ with compact support, there exists an element $u_f$ in $\mathcal{F}$ such that

$$(u, u_f) = \int u(x) f(x) dx$$

for any $u$ in $\mathcal{F}$. This element $u_f$ is called the potential generated by $f$.

Especially when $u_f(x) > 0$ p.p. in $X$ for any positive bounded measurable function $f$ with compact support, $\mathcal{F}$ is said to be positive.

Similarly as Aronszajn and Smith [1], we obtain the following lemma.

Lemma 1. Let $\mathcal{F}$ be a positive weak invariant functional space on $X$. For each $u$ in $\mathcal{F}$, there exists an element $\bar{u}$ in $\mathcal{F}$ such that

$$|u(x)| \leq \bar{u}(x) \text{ p.p. in } X \text{ and } ||u|| \leq ||\bar{u}||.$$  

Proof. Let $P$ be a closed convex cone in $\mathcal{F}$ with vertex 0 generated by the set $\{u_K \in \mathcal{F}; K \text{ is compact in } X\}$. Let $u'$ and $u''$ be the projections of $u$ and $-u$ to $P$, respectively. Put

$$\bar{u} = u' + u''.$$  

Then similarly as Aronszajn and Smith did, we see that $\bar{u}$ satisfies all the required conditions.

By the above lemma, we obtain the following

Lemma 2. Let $\mathcal{F}$ be a positive weak invariant functional space on $X$. Then $\mathcal{F}$ is a positive invariant functional space on $X$.

Proof. It is sufficient to prove that the condition (2.1) is satisfied. By Lemma 1, for any $u$ in $\mathcal{F}$,

$$\int_K |u(x)| dx \leq \int_K \bar{u}(x) dx \leq A(K) ||\bar{u}|| \leq A(K) ||u||$$

for any compact subset $K$ in $X$. Hence the condition (2.1) is satisfied and the proof is completed.

Our first theorem concerns with the converse of Deny’s theorem mentioned in the introduction.

Theorem 1. Let $X$ be a locally compact abelian group. For any positive measure $\kappa$ of positive type in $X$, there exists a positive invariant functional space with kernel $\kappa$.

Proof. By Lemma 2, it is sufficient to prove that for a positive measure $\kappa$ of positive type in $X$, there exists a positive weak invariant

3) Cf. [3], p. 209.
functional space $\mathcal{X}$ with kernel $\kappa$. Put
$\mathcal{X}' := \{\kappa * f; f \text{ is a bounded measurable function with compact support}\}$. Then $\mathcal{X}'$ is a pre-Hilbert space with norm $\|u_f\|^2 = \kappa * f * f(0)$, where $u_f := \kappa * f$ and $\check{f}(x) = f(-x)$. And we have
$$\left| \int_K u_f(x)dx \right| = |(u_f, u_{c_K})| \leq \|u_{c_K}\| \cdot \|u_f\|$$
for any compact subset $K$ in $X$, where $c_K(x)$ is the characteristic function of $K$. By the above inequality, each fundamental sequence $(u_{f_n})$ in $\mathcal{X}'$ is fundamental in the weak topology in $L^1(K)$ for any compact subset $K$ in $X$. Since $L^1(K)$ is weakly complete, there exists a function $u$ defined in $X$ such that $(u_{f_n})$ converges weakly to $u$ in $L^1(K)$ for any compact subset $K$ in $X$. Furthermore we have
$$\left| \int_K u(x)dx \right| \leq \|u_{c_K}\| \lim_{n \to \infty} \|u_{f_n}\|.$$ Let us define the norm of $u$ by
$$\|u\| = \lim_{n \to \infty} \|u_{f_n}\|.$$ Then the completion $\mathcal{X}$ of $\mathcal{X}'$ is a Hilbert space of locally summable functions and satisfies the condition (1.1) in Definition 1. We shall prove that $\mathcal{X}$ satisfies the condition (1.2). For any $x$ in $X$,
$$U_x u_f(y) = u_f(y - x) = \int f(y - x - z) d\kappa(z) = u_{v_x f}$$
for any finite continuous function $f$ with compact support. Hence for any $u$ in $\mathcal{X}$ and any $x$ in $X$,
$$U_x u \in \mathcal{X} \quad \text{and} \quad \|U_x u\| = \|u\|.$$ Thus the condition (1.2) is satisfied and the proof is completed.

2. Special Dirichlet spaces. In this section, we shall consider the kernel of a special Dirichlet space. Choquet and Deny [4] showed that a positive measure $\kappa$ of positive type is the kernel of a special Dirichlet space $D$ on a locally compact abelian group $X$ if and only if $\kappa$ is “le noyaux associé”. We give the other characterization for $\kappa$ to be the kernel of a special Dirichlet space on $X$.

Theorem 2. Let $X$ be a locally compact abelian group. A positive measure $\kappa$ of positive type in $X$ is the kernel of a special Dirichlet space $D$ on $X$ if and only if $\kappa$ satisfies the following condition ($\ast$).

($\ast$). There exists a base of compact neighborhoods $U$ of 0 such that for any $v$ in $U$, there exists a positive measure $\sigma$, satisfying that

4) Cf. [8], p. 121.
5) Cf. [3], p. 215.
6) Cf. [4], p. 4261.
(1) \[ \kappa \geq \kappa \ast \sigma_v \text{ in } X, \]
(2) \[ \kappa = \kappa \ast \sigma_v \text{ in } C_v, \]
(3) \[ \int d\sigma_v \leq 1. \]

Proof. The "only if" part follows from the existence of balayaged measures of the unit measure \( \varepsilon \) at \( O \). We shall prove the converse. For any \( v \) in \( U \), put
\[ \eta_v = \kappa \ast (\varepsilon - \sigma_v). \]

Then \( \kappa \) being symmetric,
\[ \eta_{v_1} \ast (\varepsilon - \tilde{\sigma}_{v_2}) = \eta_{v_2} \ast (\varepsilon - \tilde{\sigma}_{v_1}) \]
for any couple of \( v_1 \) and \( v_2 \) in \( U \), where the symbol \( \vee \) is the same as in the proof of Theorem 1. Hence
\[ \hat{\eta}_{v_1}(\hat{x})(1 - \tilde{\sigma}_{v_1}(\hat{x})) = \hat{\eta}_{v_2}(\hat{x})(1 - \tilde{\sigma}_{v_2}(\hat{x})) \]
in \( \hat{X} \), where the symbol \( \wedge \) over a measure represents the Fourier transform and \( \hat{X} \) is the dual group of \( X \). Put
\[ \lambda(\hat{x}) = \frac{1 - \tilde{\sigma}_{v_1}(\hat{x})}{\hat{\eta}_{v_1}(\hat{x})} = \frac{1 - \tilde{\sigma}_{v_2}(\hat{x})}{\hat{\eta}_{v_2}(\hat{x})}, \]
when \( \hat{\eta}_{v_1}(\hat{x}) \) and \( \hat{\eta}_{v_2}(\hat{x}) \) don't vanish. Then \( \lambda(\hat{x}) \) is real valued. For any \( v \) in \( U \), \( \eta_v(\hat{O}) \neq 0 \), because \( \eta_v \neq 0 \). Put
\[ \eta'_v = \eta_v \int d\eta_v. \]

Then \( \eta'_v \) converges vaguely to \( \varepsilon \) and the support of \( \eta'_v \) tends to \( \{O\} \) as \( v \) tends to \( \{O\} \). That is, \( \hat{\eta}_v(\hat{x})/\hat{\eta}_v(\hat{O}) \) converges uniformly to 1 in the wide sense. Therefore \( \lambda(\hat{x}) \) is defined everywhere in \( \hat{X} \) and
\[ \lambda(\hat{x}) = \lim_{v \to \{O\}} \frac{1 - \tilde{\sigma}_v(\hat{x})}{\hat{\eta}_v(\hat{O})}. \]

Hence \( \lambda(\hat{x}) \) is negative definite function in \( \hat{X} \), because the total mass of \( \sigma_v \) is less than or equal to 1 for any \( v \) in \( U \). By (i), \( \hat{\kappa} \) is a function defined \( p.p. \) in \( \hat{X} \) and
\[ \lambda(\hat{x})\hat{\kappa}(\hat{x}) = 1 \]
p. \( p.p. \) in \( \hat{X} \), because \( \tilde{\sigma}_v(\hat{x}) \neq 1 \) \( p.p. \) in \( \hat{X} \). That is, \( \lambda(\hat{x})^{-1} \) is locally summable. Consequently by Beurling and Deny's theorem, there exists a special Dirichlet space with kernel \( \kappa \). This completes the proof.

Remark. If \( \kappa \) is "le noyaux associé", it is obvious that \( \kappa \) satisfies the condition (*) in Theorem 2.

7) Cf. [7], Lemma 10.
8) Cf. [5], pp. 9-11.
References


